CHAPTER

Evaluation of Certain Definite Integrals & Contour Integration

(1) Integration Round The Unit Circle

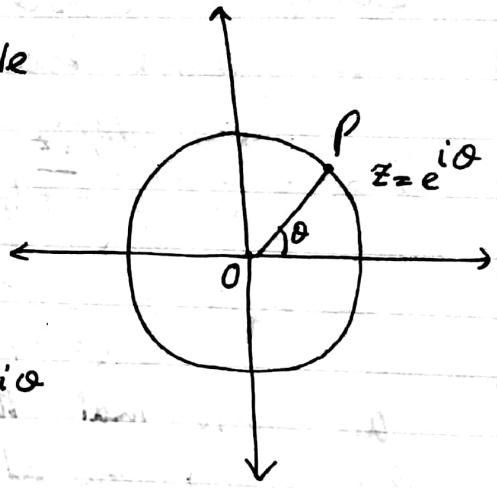
Type-I #

We ist consider integrals of the type

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration

Method # Let us draw a circle of unit radius and take its centre as origin. Consider a point P on its circumference with complex co-ordinate



$$z = e^{i\theta}$$

$$dz = ie^{i\theta} d\theta \qquad \frac{1}{z} = e^{-i\theta}$$

$$d\theta = \frac{dz}{ie^{i\theta}}$$

$$= \frac{dz}{iz}$$

Also

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

As θ ranges from 0 to 2π, the variable z ranges

once around the unit circle $|z|=1$ in the counter-clockwise sense.

$$I = \frac{1}{i}\int_C F\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{z}$$

$$= \frac{1}{i}\oint_C \frac{f(z)}{z}\,dz$$

It is notable that in case of Complex integration a curve is generally known as a Contour and the process of integration along a Contour is known a Contour integration

Example # Show by above method that

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} = 2\pi$$

Sol # $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$ $\quad d\theta = \frac{dz}{iz}$

The integral becomes

$$I = \oint_C \frac{dz/iz}{\sqrt{2}-\frac{1}{2}(z+\frac{1}{z})} = \oint_C \frac{dz}{-\frac{i}{2}(z^2-2\sqrt{2}z+1)}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$$

We note that the integrand has two simple poles, one at $z_1 = \sqrt{2}+1$ which lies outside the unit circle $C: |z|=1$ and is thus of no interest and the other at $z_2 = \sqrt{2}-1$ inside C

$$\operatorname*{Res}_{z=z_2}(f) = \left[\frac{1}{z-\sqrt{2}-1}\right]_{z=\sqrt{2}-1} = -1/2.$$

$$I = -\frac{2}{i}\cdot 2\pi i\,(-1/2) = 2\pi$$

Example # $I = \int_0^{2\pi} \frac{1}{\cos\theta + 2}\, d\theta$

Sol # $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad d\theta = \frac{dz}{iz}$

$$I = \oint_{|z|=1} \frac{-2i}{z^2 + 4z + 1}\, dz$$

$$z^2 + 4z + 1 = 0$$

$$z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm \sqrt{3}$$

$f(z) = \frac{1}{z^2+4z+1}$ has poles of order 1 at

$z_1 = -2 + \sqrt{3}$, $z_2 = -2 - \sqrt{3}$

$|z_1| = |-2 + \sqrt{3}| < 1$

$|z_2| = |2 + \sqrt{3}| > 1$

$\Rightarrow$ only z_1 lies inside the unit circle

$$\text{Res}(f, z_1) = \lim_{z \to -2+\sqrt{3}} \frac{z - (-2+\sqrt{3})}{\{z - (-2-\sqrt{3})\}\{z - (-2+\sqrt{3})\}}$$

$$= \lim_{z \to -2+\sqrt{3}} \frac{1}{z + 2 + \sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

$$I = -2i \cdot 2\pi i \cdot \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

Example # Evaluate $\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5 - 4\cos 2\theta}$

Sol # $z = e^{i\theta}$

$$\cos 3\theta = \frac{1}{2}\left(e^{3i\theta} + e^{-3i\theta}\right)$$

$$= \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$$

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$$\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$$

$$d\theta = \frac{dz}{iz}$$

$$I = \int_C \frac{\left(z^3 + \frac{1}{z^3}\right)^2 dz}{4i\left\{5 - 2\left(z^2 + \frac{1}{z^2}\right)\right\}}$$

$$= \int_C \frac{(z^6+1)^2/z^6}{4i\left\{\frac{5z^2 - 2z^4 + 1}{z^2}\right\}} \frac{dz}{z}$$

$$= \int_C \frac{(z^6+1)^2\, dz}{-8i\, z^5\left(2z^4 - 5z^2 + 2\right)}$$

$2z^4 - 5z^2 + 2 = 0$

$$= \int_C \frac{(z^6-1)^2\, dz}{-8i z^5 (z^2-2)(2z^2-1)}$$

$2z^4$

$$z^2 = \frac{5 \pm \sqrt{25+16}}{4}$$

$$z^2 = 2$$

$\Rightarrow 2\cdot 4 - 10 = 0$

$2z^4 - 4z^2 - z^2 + 2 = 0$

$2z^2(z^2-2) - 1(z^2-2) = 0$

$$= \int_C \frac{(z^6-1)^2\, dz}{-8i z^5 (z-\sqrt{2})(z+\sqrt{2})\left(z-\frac{1}{\sqrt{2}}\right)\left(z+\frac{1}{\sqrt{2}}\right)}$$

$$f(z) = \frac{(z^6-1)^2}{-4iz^5(z^2-2)(2z^2-1)}$$

has a pole of order 5 at $z=0$ and simple poles at $z = \pm\frac{1}{\sqrt{2}}$ inside the contour and at $z = \pm\sqrt{2}$ outside the contour

For residue at $z=0$

$$f(z) = -\frac{(1+z^6)^2}{8iz^5}\left(1 - \frac{z^2}{2}\right)^{-1}(1-2z^2)^{-1}$$

$$= -\frac{1}{8iz^5}\left(1 + 2z^6 + z^{12}\right)\left(1 + \frac{z^2}{2} + \frac{z^4}{4} + \cdots\right)$$

$$\times\left(1 + 2z^2 + 4z^4 + \cdots\right)$$

Co-efficient of $\frac{1}{z} = \frac{-21}{32i}$

$$R\left(f, \frac{1}{\sqrt{2}}\right) = \left[\frac{-(z^6+1)^2}{8iz^5(z+\frac{1}{\sqrt{2}})(z^2-2)}\right]_{z=\frac{1}{\sqrt{2}}} = 27/64i$$

Similarly
$R(f, -\frac{1}{2}) = 27/64i$
The sum of residues $= -\frac{21}{32i} + \frac{27}{64i} + \frac{27}{64i} = \frac{3}{16i}$

$$I = 2\pi i \times \frac{3}{16i} = \frac{3\pi}{8}$$

Example # Prove that

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$$

Sol # Choosing the contour as a circle of unit radius ($|z|=1$)

$$z = e^{i\theta} \quad \therefore dz = ie^{i\theta}d\theta$$
$$d\theta = \frac{dz}{iz} \quad \text{where } \theta \text{ varies from } 0 \text{ to } 2\pi$$

$$z + \frac{1}{z} = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

Thus
$$I = \int_C \frac{dz}{iz\left\{2+\frac{1}{2}\left(z+\frac{1}{z}\right)\right\}} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}$$

where C denotes the unit circle $|z|=1$
Poles of the integrand will be given by

$$z^2+4z+1=0$$

or $$z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm 3$$

of which the pole $z = -2-\sqrt{3}$ lies outside the contour and therefore the only pole that lies within the contour is $z = -2+\sqrt{3}$ which is of the order of one.
Residue at $z = -2+\sqrt{3}$

$$= \lim_{z \to -2+\sqrt{3}} \{z-(-2+\sqrt{3})\} f(z)$$

$$= \frac{2}{i} \lim_{z \to -2+\sqrt{3}} (z+2-\sqrt{3}) \frac{1}{(z^2+4z+1)}$$

$$= \frac{2}{i} \lim_{z \to -2+\sqrt{3}} \left(\frac{1}{z+2+\sqrt{3}} \right)$$

$$= \frac{2}{i} \cdot \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{i\sqrt{3}}$$

$$I = 2\pi i \times \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

Example # Prove that $I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta}$

Sol # Choosing the contour C as a circle $|z| = 1$ of unit radius

$$z = e^{i\theta} \qquad d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$I = \int_C \frac{\left\{\frac{1}{2i}\left(z - \frac{1}{z}\right)\right\}^2}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{z}$$

$$= -\frac{1}{i} \int_C \frac{\left(z - \frac{1}{z}\right)^2 dz}{2z\left\{2a + b\left(z + \frac{1}{z}\right)\right\}}$$

$$= -\frac{1}{2i} \int_C \frac{(z^2-1)^2 \, dz}{z^2(2az + bz^2 + b)}$$

$$= -\frac{1}{2ib} \int_C \frac{(z^2-1)^2 dz}{z^2\left(z^2 + \frac{2a}{b}z + 1\right)}$$

$$= -\frac{1}{2ib} \int_C \frac{(z^2-1)^2 dz}{z^2(z-\alpha)(z-\beta)}$$

where $\alpha = \frac{-a + \sqrt{a^2-b^2}}{b}$ $\qquad \beta = \frac{-a - \sqrt{a^2-b^2}}{b}$

are the roots of the quadratic $z^2 + \frac{29}{6}z + 1 = 0$

So $\alpha + \beta = -\frac{29}{6}$ $\quad \alpha\beta = 1$

$\Rightarrow |\alpha|\,|\beta| = 1$ $\quad$ where $|\beta| > |\alpha|$

$\therefore |\beta| = \frac{1}{|\alpha|}$

$\Rightarrow \frac{1}{|\alpha|} > |\alpha| \Rightarrow |\alpha|^2 < 1$

$\Rightarrow |\alpha| < 1$

So $z = \alpha$ is the only simple pole within the contour C, $z = \beta$ lies outside the contour. Also $z = 0$ is a pole of order two which lies inside the contour.

Residue at $z = \alpha = \lim_{z \to \alpha} (z - \alpha)\left(\frac{-1}{2ib}\right) \frac{(z^2-1)^2 dz}{z^2(z-\alpha)(z-\beta)}$

$$= -\frac{1}{2ib} \lim_{z \to \alpha} \frac{(z^2-1)^2}{z^2(z-\beta)}$$

$$= -\frac{1}{2ib} \lim_{z \to \alpha} \frac{\left(z - \frac{1}{z}\right)^2}{z - \beta}$$

$$= -\frac{1}{2ib} \frac{\left(\alpha - \frac{1}{\alpha}\right)^2}{\alpha - \beta} = \frac{(\alpha - \beta)^2}{-2ib(\alpha - \beta)} \quad \because \beta = \frac{1}{\alpha}$$

$$= \frac{\alpha - \beta}{-2ib} = \frac{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}{-2ib}$$

$$= -\frac{1}{2ib} \cdot \frac{2}{b}\sqrt{a^2 - b^2}$$

$$= \frac{\sqrt{a^2 - b^2}}{-ib^2}$$

Residue at $z = 0$ is the co-efficient of $\frac{1}{z}$ in $-\frac{1}{2ib} \frac{(z^2-1)^2}{z^2\left(z^2 + 1 + \frac{29}{6}z\right)}$, where z is a small quantity near pole

$$\text{Now } -\frac{1}{2ib} \frac{(z^2-1)^2}{z^2\left(z^2 + 1 + \frac{29}{6}z\right)} = -\frac{1}{2ib} \cdot \frac{\left(1 - \frac{1}{z^2}\right)^2}{\left(1 + \frac{29}{6}\frac{1}{z} + \frac{1}{z^2}\right)}$$

$$= -\frac{1}{2ib}\left(1 - \frac{1}{z^2}\right)^2 \left(1 + \frac{29}{6z} + \frac{1}{z^2}\right)^{-1}$$

$$= -\frac{1}{2ib}\left(1-\frac{2}{z^2}+\frac{1}{z^4}\right)\left(1-\left(\frac{2a}{bz}+\frac{1}{z^2}\right)+\cdots\right)$$

Residue at $z=0 = -\frac{1}{2ib}\left(-\frac{2a}{b}\right) = \frac{a}{ib^2}$

$$I = 2\pi i\left\{-\frac{\sqrt{a^2-b^2}}{ib^2}+\frac{a}{ib^2}\right\} = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

Example # Evaluate $I = \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta$

$= \frac{2\pi}{\lfloor n}$, where n is a +ve integer

Sol # $|z|=1$ Choosing the contour C as a unit circle

$$z = e^{i\theta} \qquad \frac{1}{z} = e^{-i\theta}$$

$$d\theta = \frac{dz}{iz}$$

Consider

$$J = \int_0^{2\pi} e^{\cos\theta}\cdot e^{i(\sin\theta - n\theta)}\cdot d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta + i\sin\theta}\cdot\left(e^{-i\theta}\right)^n d\theta$$

I is real part in J

$$J = \int_0^{2\pi} e^{e^{i\theta}}\left(e^{-i\theta}\right)^n d\theta$$

on putting $z = e^{i\theta}$ $\quad e^{-i\theta} = \frac{1}{z}$ $\quad d\theta = \frac{dz}{iz}$

$$J = \int_0^{2\pi} e^{z}\cdot\frac{1}{z^n}\cdot\frac{dz}{iz}$$

Here the only pole inside the contour C is $z=0$ of order $n+1$

$$J = \frac{1}{i}\int_C \frac{e^z}{z^{n+1}}\,dz$$

Residue at $z=0$ $= \frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}\left\{(z-0)^{n+1}\cdot\frac{e^z}{z^{n+1}}\cdot\frac{1}{i}\right\}$

$$= \frac{1}{i}\lim_{z\to 0}\frac{d^n}{dz^n}e^z\cdot\frac{1}{n!}$$

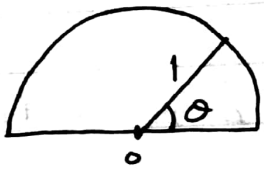
$$= \frac{1}{i}\cdot\frac{1}{n!}\lim_{z\to 0}e^z = \frac{1}{i}\cdot\frac{1}{n!}$$

$$J = 2\pi i\cdot\frac{1}{i\,n!} = \frac{2\pi}{n!}$$

$$I = \text{Real part of } J = \int_0^{2\pi} e^{\cos\alpha}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}$$

Example # Evaluate $I = \int_0^{\pi}\frac{a\,d\theta}{a^2+\sin^2\theta}$

Sol # Choosing the contour C as a semi-circle of unit radius



$z = e^{i\theta}$ $\quad d\theta = \frac{dz}{iz}$

$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$$I = \frac{a}{i}\int_C \frac{1}{a^2+\left\{\frac{1}{2i}\left(z-\frac{1}{z}\right)\right\}^2}\cdot\frac{dz}{iz}$$

$$= \frac{4a}{i}\int_C \frac{z\,dz}{z^2(4a^2+2)-z^4-1}$$

$$= -\frac{4a}{i}\int_C \frac{dz}{z^4-z^2(4a^2+2)+1}$$

poles are given by $z^4 - z^2(4a^2+2)+1 = 0$

Let us assume that

$$z^4 - z^2(4a^2+2)+1 = (z^2-\alpha^2)(z^2-\beta^2)$$

$\therefore\ \alpha^2+\beta^2 = 4a^2+2$ & $\alpha^2\beta^2 = 1$

If $\alpha < 1$, then $\alpha^2 < 1 \Rightarrow \frac{1}{\alpha^2} > 1$

$\rightarrow \beta^2 > 1$

$\rightarrow \beta > 1$

$\Rightarrow$ pole within the semi-circle will be $z = \alpha$ and $z = \beta$ will lie outside the contour

Residue at $z = \alpha$

$$= \lim_{z \to \alpha} (z - \alpha) \left(-\frac{4q}{i}\right) \frac{z}{(z^2 - \alpha^2)(z^2 - \beta^2)}$$

$$= -\frac{4q}{i} \lim_{z \to \alpha} \frac{z}{(z + \alpha)(z^2 - \beta^2)}$$

$$= -\frac{4q}{i} \frac{\alpha}{2\alpha(\alpha^2 - \beta^2)}$$

$$= -\frac{2q}{i(\alpha^2 - \beta^2)} = \frac{2q}{i(\beta^2 - \alpha^2)}$$

$$I = 2\pi i \times \frac{2q}{i(\beta^2 - \alpha^2)}$$

where $\beta^2 - \alpha^2 = \sqrt{(\beta^2 + \alpha^2)^2 - 4\alpha^2\beta^2}$

$$= \sqrt{(4a^2 + 2)^2 - 4}$$

$$= 4a\sqrt{1 + a^2}$$

$$I = \frac{4\pi q}{4a\sqrt{1 + a^2}} = \frac{\pi}{\sqrt{1 + a^2}}$$

If $\beta < 1$, then $\beta^2 < 1 \Rightarrow \frac{1}{\beta^2} > 1$

$\Rightarrow \alpha^2 > 1$

$\Rightarrow \alpha > 1$

The pole $z = \beta$ lies inside the contour

Residue at $z = \beta$ $= \lim_{z \to \beta} (z - \beta) \left(-\frac{4q}{i}\right) \frac{z}{(z^2 - \alpha^2)(z^2 - \beta^2)}$

$$= -\frac{4q}{i} \cdot \frac{\beta}{2\beta(\beta^2-\alpha^2)}$$

$$= -\frac{2q}{i} \cdot \frac{1}{\beta^2-\alpha^2}$$

$$= \frac{2q}{i} \cdot \frac{1}{\alpha^2-\beta^2}$$

$$\alpha^2-\beta^2 = \sqrt{(\alpha^2+\beta^2)^2-4\alpha^2\beta^2}$$

$$= \sqrt{(4a^2+2)^2-4} = 4a\sqrt{1+a^2}$$

Residue

$$= \frac{2q}{i} \cdot \frac{1}{4a\sqrt{1+a^2}}$$

$$I = 2\pi i \cdot \frac{2q}{i} \cdot \frac{1}{4a\sqrt{1+a^2}}$$

$$= \frac{\pi}{\sqrt{1+a^2}}$$

OR

$$I = \int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \int_0^{\pi} \frac{2a\,d\theta}{2a^2+2\sin^2\theta}$$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+(1-\cos 2\theta)}$$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+1-\cos 2\theta}$$

Let $2\theta = \phi$ when $\theta = 0$ $\phi = 0$

$2d\theta = d\phi$ $\theta = \pi$ $\phi = 2\pi$

$2a\,d\theta = a\,d\phi$

$$I = \int_0^{2\pi} \frac{a\,d\phi}{2a^2+1-\cos\phi}$$

Let $z = e^{i\theta}$ $dz = i e^{i\theta} d\theta$ $\Rightarrow d\theta = \frac{dz}{iz}$

$$\cos\varphi = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$I = \int_C \frac{a\,dz}{iz\left[2a^2+1-\frac{1}{2}\left(z+\frac{1}{z}\right)\right]}$$ where C is the unit circle

$$= \int_C \frac{a\,dz}{iz\left[\frac{2(2a^2+1) - z - \frac{1}{z}}{2}\right]}$$

$$= \int_C \frac{2a\,dz}{iz\left[\frac{2(2a^2+1)z - z^2 - 1}{z}\right]}$$

$$= \frac{2a}{i}\int_C \frac{dz}{-z^2+2(2a^2+1)z-1}$$

$$= -\frac{2a}{i}\int_C \frac{dz}{z^2-2(2a^2+1)z+1}$$

$$= 2ai\int_C \frac{dz}{z^2-2(2a^2+1)z+1}$$

$$f(z) = \frac{1}{z^2-2(2a^2+1)z+1}$$

poles of $f(z)$ are the roots of

$$z^2 - 2z(2a^2+1) + 1 = 0$$

$$z = \frac{2(2a^2+1) \pm \sqrt{4(2a^2+1)^2-4}}{2}$$

$$= \frac{2(2a^2+1) \pm 2\sqrt{(2a^2+1)^2-1}}{2}$$

$$= \frac{(2a^2+1) \pm \sqrt{4a^4+4a^2+1-1}}{2}$$

$$= \frac{(2a^2+1) \pm \sqrt{4a^4+4a^2}}{2}$$

$$= (2a^2+1) \pm 2a\sqrt{a^2+1}$$

Let $\alpha = (2a^2+1) + 2a\sqrt{a^2+1}$

$\beta = (2a^2+1) - 2a\sqrt{a^2+1}$

Clearly $|\alpha| > 1$

Now $\alpha\beta = 1$

$\Rightarrow |\alpha||\beta| = 1$

$|\beta| = \frac{1}{|\alpha|}$

$\frac{1}{|\alpha|} < 1 \quad \Rightarrow |\beta| < 1$

Hence the only pole inside C i.e unit circle is $z=\beta$

$$R(f,\beta) = \lim_{z\to\beta} \frac{(z-\beta)}{(z-\alpha)(z-\beta)} = \frac{1}{\beta-\alpha} = -\frac{1}{\alpha-\beta}$$

$$= \frac{-1}{\sqrt{(\beta+\alpha)^2 - 4\alpha\beta}}$$

$$= \frac{-1}{\sqrt{\{2(2a^2+1)\}^2 - 4\cdot 1}}$$

$$= \frac{-1}{\sqrt{4(2a^2+1)^2 - 4}}$$

$$= \frac{-1}{2\sqrt{4a^4+4+4a^2-4}}$$

$$= \frac{-1}{4a\sqrt{a^2+1}}$$

$$I = 2ai \times 2\pi i \times \frac{-1}{4a\sqrt{a^2+1}}$$

$$= \frac{\pi}{\sqrt{a^2+1}}$$

Example # Evaluate $I = \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1 - 2p\cos 2\theta + p^2}$

$(0 < p < 1)$

Sol $I = \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1 - 2p\cos 2\theta + p^2}$

$2\cos 3\theta = z^3 + \frac{1}{z^3}$ $\qquad 2\cos 2\theta = z^2 + \frac{1}{z^2}$.

$d\theta = \frac{dz}{iz}$ $\qquad z = e^{i\theta}$

$$I = \frac{1}{i} \int_C \frac{\left\{\frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)\right\}^2 dz/z}{1 - p\left(z^2 + \frac{1}{z^2}\right) + p^2}$$

$$= \frac{1}{i} \int_C \frac{(z^6+1)^2 z^2}{4z^6\{(1+p^2)z^2 - pz^4 - p\}} \cdot \frac{dz}{z}$$

$$= -\frac{1}{4i} \int_C \frac{(z^6+1)^2}{z^5[pz^4 - (1+p^2)z + p]}$$

$$= -\frac{1}{4i} \int_C \frac{(z^6+1)^2 \, dz}{z^5(pz^2-1)(z^2-p)}$$

OR

$$I = \frac{1}{2} \int_0^{2\pi} \frac{2\cos^2 3\theta \, d\theta}{1 - 2p\cos^2\theta + p^2}$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{(1 + \cos 6\theta)\, d\theta}{1 - 2p\cos 2\theta + p^2}$$

$$= \text{Real part of } \frac{1}{2} \int_0^{2\pi} \frac{(1 + e^{6i\theta})\, d\theta}{1 - 2p\cos 2\theta + p^2}$$

Let $z = e^{i\theta}$ $\quad z^2 = e^{2i\theta}$

$\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$ $\quad d\theta = \frac{dz}{iz}$

$$I = \frac{1}{2}\int_C \frac{1+z^6}{1-p\left(z^2+1/z^2\right)+p^2}\cdot\frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z(1+z^6)\,dz}{(1-pz^2)(z^2-p)}$$

$$= \frac{1}{2i}\int_C f(z)\,dz$$

$f(z)$ has simple poles at $z = \pm 1/\sqrt{p}$ and $z = \pm\sqrt{p}$

The only poles which lie inside the unit circle are

$$z = \pm\sqrt{p} \qquad (0 < p < 1)$$

$$R_1(f, \sqrt{p}) = \lim_{z\to\sqrt{p}}\left[\frac{z(1+z^6)}{(1-pz^2)(z+\sqrt{p})}\right]$$

$$= \frac{\sqrt{p}\,(1+p^3)}{(1-p^2)\,2\sqrt{p}} = \frac{1+p^3}{2(1-p^2)}$$

$$R_2(f, -\sqrt{p}) = \frac{-\sqrt{p}\,(1+p^3)}{(1-p^2)(-2\sqrt{p})} = \frac{1+p^3}{2(1-p^2)}$$

$$\Sigma R = R_1 + R_2 = \frac{1+p^3}{2(1-p^2)} + \frac{1+p^3}{2(1-p^2)}$$

$$= \frac{2(1+p^3)}{2(1-p^2)}$$

$$= \frac{1+p^3}{1-p^2}$$

$$\frac{1}{2}\int_0^{2\pi} \frac{1+e^{6i\theta}}{1-2p\cos 2\theta+p^2}\,d\theta = 2\pi i\cdot\frac{1}{2i}\times\frac{1+p^3}{1-p^2} \quad \underline{16}$$

$$= \pi\,\frac{(1+p)(1-p+p^2)}{(1+p)(1-p)}$$

Equating Real parts on both sides

$$\int_0^{2\pi} \frac{\cos^2 3\theta\,d\theta}{1-2p\cos 2\theta+p^2} = \frac{\pi(1-p+p^2)}{1-p} \quad (0<p<1)$$

Example # Evaluate $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$

Sol # Choosing the contour C as a circle of unit radius $|z|=1$

$$z = e^{i\theta} \qquad d\theta = \frac{dz}{iz}$$

$$I = \int_0^{2\pi} \frac{2\cos^2\theta - 1}{5+4\cos\theta}\,d\theta$$

$$= \int_C \frac{\frac{2}{4}\left(z+\frac{1}{z}\right)^2 - 1}{5+2\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz}$$

$$= \int_C \frac{\frac{1}{2}\left(z^4+2+\frac{1}{z^2}\right)-1}{\frac{5z^2+2z^2+2}{z}}\cdot\frac{dz}{iz}$$

$$= \int_C \frac{2z^2-z+2}{2z^2+5z+2}\cdot\frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(2z^2+5z+2)}\,dz$$

$$= \frac{1}{2i}\int_C \frac{z^4+1}{z^2(2z+1)(z+2)}\,dz \qquad \text{17}$$

within C are a simple pole at $z = -1/2$ and pole of order 2 at $z=0$

Residue at $z=-1/2 = \lim_{z\to -1/2}(z+\frac{1}{2})\frac{1}{2i}\frac{z^4+1}{z^2(2z+1)(z+2)}$

$$= \frac{17}{24}i$$

Residue at $z=0$ = Co-efficient of $\frac{1}{z}$ in $\frac{1}{2i}\frac{z^4+1}{z^2(2z+1)(z+2)}$

Now

$$\frac{1}{2i}\frac{z^4+1}{z^2(2z+1)(z+2)} = \frac{1}{4i}\left(1+\frac{1}{2z}\right)\left(1+\frac{1}{2z}\right)^{-1}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{4i}\left(1+\frac{1}{2z}\right)\left(1-\frac{1}{2z}+\cdots\right)\left(1-\frac{2}{z}+\cdots\right)$$

$$= \frac{1}{4i}\left(-5/2\right) = -\frac{5}{8i}$$

$I = 2\pi i \times$ Sum of residues

$$= 2\pi i\left[\frac{17}{24i} - \frac{5}{8i}\right] = \frac{\pi}{6}$$

<u>Example</u> # Evaluate the integral $\int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos n\theta}{3+2\cos\theta}\,d\theta$

<u>Sol</u>

$$I = \int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos n\theta}{3+2\cos\theta}\,d\theta$$

$$= \text{Real part of}\int_0^{2\pi}\frac{(1+2\cos\theta)\,e^{in\theta}}{3+2\cos\theta}\,d\theta$$

$$I_1 = \int_0^{2\pi}\frac{(1+2\cos\theta)\,e^{in\theta}}{3+2\cos\theta}\,d\theta$$

Let $z = e^{i\theta}$ $\qquad d\theta = \frac{dz}{iz}$

$$I_1 = \int_C \frac{\left(1+z+\frac{1}{z}\right)^n z^n}{\left(3+z+\frac{1}{z}\right)} \frac{dz}{iz}$$

where C is a unit circle $|z|=1$

$$= \frac{1}{i}\int_C \frac{(1+z+z^2)^n}{(1+3z^2+z^2)}\,dz$$

$$f(z) = \frac{(1+z+z^2)^n}{(1+3z+z^2)}$$

poles of f are at roots of $z^2+3z+1=0$

$$\alpha = \frac{-3+\sqrt{5}}{2} \qquad \beta = \frac{-3-\sqrt{5}}{2}$$

$$\alpha\beta = 1$$

$|\beta| > |\alpha| \qquad |\alpha| < 1 \qquad |\beta| > 1$

$\Rightarrow$ pole $z=\alpha$ lies within C

$$R(f,\alpha) = \lim_{z\to\alpha} \frac{(z-\alpha)\cdot(1+z+z^2)^n}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{(1+\alpha+\alpha^2)^n}{\alpha-\beta}$$

$$= \frac{\left(1-\frac{3}{2}+\frac{\sqrt{5}}{2}+\frac{7-3\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$$= \frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

$$I_1 = 2\pi i \times \frac{1}{i} \times \frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

$$= \frac{2\pi\,(3-\sqrt{5})^n}{\sqrt{5}}$$

Equating real parts of both sides

$$I = \int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta}\,d\theta = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

Example # Prove that

(a) $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ $\quad a>b>0$

(b) $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ $\quad a>b>0$

(c) Also deduce the values of integals

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} \quad \& \quad \int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2}$$

Sol (a) $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$

Let $z = e^{i\theta}$ $\qquad d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$I = \int_C \frac{1}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b}$$

where C is the unit circle $|z|=1$

$$f(z) = \frac{1}{bz^2+2az+b}$$

The poles of $f(z)$ are roots of

$bz^2+2az+b=0$

$$z = -\frac{a \pm \sqrt{a^2-b^2}}{b}$$

$$z_1 = -\frac{a+\sqrt{a^2-b^2}}{b} \qquad z_2 = -\frac{a-\sqrt{a^2-b^2}}{b}$$

Since $a>b>0$

$\therefore |z_2|>1$ $\qquad z_1 z_2 = 1$ $\qquad \Rightarrow |z_1||z_2| = 1$

$\Rightarrow \frac{1}{|z_1|}>1 \quad \Rightarrow |z_1|<1$

$\Rightarrow z_1$ lies within C

$$R(f, z_1) = \lim_{z \to z_1} \left[\frac{(z - z_1)}{b(z - z_1)(z - z_2)} \right] \quad \text{(20)}$$

$$= \frac{1}{b(z_1 - z_2)}$$

$$= \frac{1}{b\sqrt{(z_1 + z_2)^2 - 4 z_1 z_2}}$$

$$= \frac{1}{b\sqrt{\left(-\frac{2q}{b}\right)^2 - 4}}$$

$$= \frac{1}{b\sqrt{\frac{4a^2 - 4b^2}{b}}} = \frac{1}{2\sqrt{(a^2 - b^2)}}$$

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2}{i} \times 2\pi i \times \frac{1}{2\sqrt{a^2 - b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2 - b^2}}$$

(b) Similarly can be proved.

(c) Deduction

We have

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Diff. w.r.t a

$$\int_0^{2\pi} \frac{-1\, d\theta}{(a + b\cos\theta)^2} = 2\pi\left(-\tfrac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot 2a$$

$$= -\frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

Also we have

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Diff. w.r.t a

$$\int_0^{2\pi} \frac{d\theta}{(a + b\sin\theta)^2} = -\frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

Remarks # $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} \overset{21}{=} 2\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2}$

$\therefore \int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$

putting $b=1$

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}} \qquad a>0$$

Example # (a) prove that $\int_0^{2\pi} \frac{d\theta}{a+b\cos^2\theta} = \frac{2\pi}{\sqrt{a(a+b)}}$

(b) prove that $\int_0^{2\pi} \frac{d\theta}{a+b\sin^2\theta} =$

Hence deduce the value of integrals

$\int_0^{2\pi} \frac{d\theta}{(a+b\cos^2\theta)^2}$ & $\int_0^{2\pi} \frac{d\theta}{(a+b\sin^2\theta)^2}$

Sol # (a) $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos^2\theta}$

$= \int_0^{2\pi} \frac{d\theta}{a+(b/2)(1+\cos 2\theta)}$

$= \int_0^{2\pi} \frac{2\,d\theta}{2a+b+b\cos 2\theta}$

Let $2\theta = \varphi$ when $\theta = 0$ $\varphi = 0$

$\theta = 2\pi$ $\varphi = 4\pi$

$$I = \int_0^{4\pi} \frac{2\,d\varphi}{2a+b+b\cos\varphi}$$

$$= 2\int_0^{2\pi} \frac{d\varphi}{2a+b+b\cos\varphi}$$

Let $z = e^{i\varphi}$ $\cos\varphi = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$d\varphi = \frac{dz}{iz}$$

$$I = 2\int_C \frac{\frac{dz}{iz}}{2a+b+\frac{b}{2}\left(z+\frac{1}{z}\right)}$$

$$= \frac{4}{i}\int_C \frac{dz}{bz^2+2(2a+b)z+b}$$

$$= \frac{4}{ib}\int_C \frac{dz}{z^2+\frac{2(2a+b)}{b}z+1}$$

$$= \frac{4}{ib}\int_C f(z)\,dz$$

poles of $f(z)$ are roots of $z^2+\frac{2(2a+b)}{b}+1=0$

$$z = -\frac{\frac{2}{b}(2a+b) \pm \sqrt{\frac{4}{b^2}(2a+b)^2-4}}{2}$$

$$= -\frac{(2a+b) \pm \sqrt{4a^2+4ab}}{b}$$

$$= -\frac{(2a+b) \pm 2\sqrt{a^2+ab}}{b}$$

$$z_1 = -\frac{(2a+b)+2\sqrt{a(a+b)}}{b} \qquad z_2 = -\frac{(2a+b)-2\sqrt{a(a+b)}}{b}$$

$z_1 z_2 = 1$ $\qquad |z_2| > 1$

$|z_1||z_2| = 1$

$|z_2| = \frac{1}{|z_1|}$ $\qquad \frac{1}{|z_2|} < 1$

$\Rightarrow |z_1| < 1$

The only pole inside C is $z = z_1$

$$R(f, z_1) = \lim_{z\to z_1} (z-z_1)\cdot\frac{1}{(z-z_1)(z-z_2)} = \frac{1}{z_1-z_2}$$

$$= \frac{b}{-(2a+b) + 2\sqrt{a(a+b)} + (2a+b) + 2\sqrt{a(a+b)}}$$

$$= \frac{b}{4\sqrt{a(a+b)}}.$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos^2\theta} = \frac{4}{ib} \times 2\pi i \times \frac{b}{4\sqrt{a(a+b)}}$$

$$= \frac{2\pi}{\sqrt{a(a+b)}}$$

(b) $$I = \int_0^{2\pi} \frac{d\theta}{a+b\sin^2\theta}$$

$$= \int_0^{2\pi} \frac{d\theta}{a + \frac{b}{2}(1-\cos 2\theta)}$$

$$= \int_0^{4\pi} \frac{d\phi/2}{a+\frac{b}{2}(1-\cos\phi)} \qquad \text{let } 2\theta = \phi$$

$$= 2\int_0^{2\pi} \frac{d\phi}{2a+b-b\cos\phi}$$

Let $z = e^{i\phi}$ $\qquad d\phi = \frac{dz}{iz}$

$$\cos\phi = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$I = 2\int_C \frac{dz/iz}{2a+b-\frac{b}{2}\left(z+\frac{1}{z}\right)}$$

$$= \frac{4}{i}\int_C \frac{dz}{2(2a+b)z - bz^2 - b}.$$

$$= -\frac{4}{ib}\int_C \frac{dz}{z^2 - \frac{2}{b}(2a+b) + 1}$$

$$f(z) = \frac{1}{z^2 - \frac{2}{b}(2a+b)z + 1} \overset{24}{=}$$

The roots of $z^2 - \frac{2}{b}(2a+b)z + 1 = 0$ are poles of $f(z)$

$$z = \frac{\frac{2}{b}(2a+b) \pm \sqrt{\frac{4}{b^2}(2a+b)^2 - 4}}{2}$$

$$= \frac{(2a+b) \pm \sqrt{4a^2 + 4ab}}{b}$$

$$= \frac{(2a+b) \pm 2\sqrt{a(a+b)}}{b}$$

$$z_1 = \frac{(2a+b) + 2\sqrt{a(a+b)}}{b}$$

$$z_2 = \frac{(2a+b) - 2\sqrt{a(a+b)}}{b}$$

$|z_1| > 1$ Also $z_1 z_2 = 1$

$|z_1||z_2| = 1$

$\frac{1}{|z_1|} < 1$ $|z_2| = \frac{1}{|z_1|}$

$|z_2| < 1$

$\Rightarrow$ z_2 lies inside the contour C

$$R(f, z_2) = \lim_{z \to z_2} \frac{z - z_2}{(z - z_1)(z - z_2)}$$

$$= \frac{1}{z_2 - z_1}$$

$$= \frac{b}{(2a+b) - 2\sqrt{a(a+b)} - (2a+b) - 2\sqrt{a(a+b)}}$$

$$= \frac{b}{-4\sqrt{a(a+b)}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin^2\theta} = -\frac{4}{ib} \times 2\pi i \times \frac{b}{-4\sqrt{a(a+b)}}$$

$$= \frac{2\pi}{\sqrt{a(a+b)}}$$

Deductions

We have

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos^2\theta} = \frac{2\pi}{\sqrt{a^2+ab}} = 2\pi (a^2+ab)^{-1/2}$$

Diff. w.r.t a

$$\int_0^{2\pi} \frac{-1\, d\theta}{(a+b\cos^2\theta)^2} = -\pi (a^2+ab)^{-3/2} (2a+b)$$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos^2\theta)^2} = \frac{\pi(2a+b)}{a^{3/2}(a+b)^{3/2}}$$

Also

$$\int_0^{2\pi} \frac{d\theta}{(a+b\sin^2\theta)} = \frac{2\pi}{\sqrt{a^2+ab}}$$

Diff w.r.t a

$$\int_0^{2\pi} \frac{d\theta}{(a+b\sin^2\theta)^2} = \frac{\pi(2a+b)}{a^{3/2}(a+b)^{3/2}}$$

Remarks #

$$\int_0^{2\pi} \frac{d\theta}{(a+b\sin^2\theta)^2} = 2\int_0^{\pi} \frac{d\theta}{(a+b\sin^2\theta)^2}$$

$$\Rightarrow \int_0^{\pi} \frac{d\theta}{(a+b\sin^2\theta)^2} = \frac{\pi(2a+b)}{2a^{3/2}(a+b)^{3/2}}$$

$$\& \int_0^{\pi} \frac{d\theta}{(a+b\cos^2\theta)^2} = \frac{\pi(2a+b)}{2a^{3/2}(a+b)^{3/2}}$$

Example Prove that $\int_0^{2\pi} \cos^{2n}\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot 2\pi$ where $n = 1, 2, 3, \cdots$

Sol

$$I = \int_0^{2\pi} \cos^{2n}\theta \, d\theta$$

Let $z = e^{i\theta}$ $\quad dz = iz\,d\theta$

$$d\theta = \frac{dz}{iz}$$

$$I = \int_0^{2\pi} \cos^{2n}\theta \, d\theta = \int_C \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^{2n} \frac{dz}{iz}$$

$$= \frac{1}{2^{2n} i} \int_C \frac{(z^2+1)^{2n}}{z^{2n+1}} \cdot dz$$

$$f(z) = \frac{(z^2+1)^{2n}}{z^{2n+1}}$$

has a pole of order $2n+1$ at $z = 0$

$$f(z) = \frac{(z^2+1)^{2n}}{z^{2n+1}} = \frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}$$

$$= \frac{1}{z}\left[z^{2n} + \binom{2n}{1} z^{2n-1}\left(\frac{1}{z}\right) + \cdots \binom{2n}{n} z^n \cdot \frac{1}{z^n} + \cdots + \binom{2n}{2n}\left(\frac{1}{z}\right)^{2n} \right]$$

Res $(f, 0)$ = Co-efficient of $\frac{1}{z} = \binom{2n}{n}$

$$I = \frac{1}{2^{2n} i} \times 2\pi i \times \binom{2n}{n}$$

$$= \frac{1}{2^{2n}} \cdot 2\pi \cdot \frac{(2n)!}{n!\, n!}$$

$$= \frac{1\cdot 2 \cdots n(n-1) \cdots (2n-2)\cdot 2n}{2^{2n} \cdot n!\, n!} \cdot 2\pi$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2^n} \cdot 2\pi$$

II Improper Integrals of Rational Functions

Form $\int_{-\infty}^{+\infty} f(x)\,dx$ OR $\int_{0}^{\infty} f(x)\,dx$.

Where $f(x)$ is a rational function

Consider real integral of Type

$$I = \int_{-\infty}^{+\infty} f(x)\,dx \longrightarrow (A)$$

Such an integral for which the interval of integration is not finite, is called an improper integral and it has meaning

$$I = \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{0} f(x)\,dx + \lim_{b\to\infty}\int_{0}^{b} f(x)\,dx \longrightarrow (B)$$

$$= \lim_{a\to-\infty}\int_{a}^{\alpha} f(x)\,dx + \lim_{b\to\infty}\int_{\alpha}^{b} f(x)\,dx.$$

Integral $\int_{-\infty}^{+\infty} f(x)\,dx$ exists if both limits on the R.H.S exist.

It is useful to consider a more restrictive limit and consider the integral

$$I_p = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \longrightarrow (C)$$

It is called the Cauchy principal value of the

the integral

It may exist even if the limits in Ⓑ do not exist e.g

If $f(x)$ is odd and non-zero at ∞ (e.g $f(x) = x$), then $I_p = 0$ but I will not exist.

$$\lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{R^2}{2} \right) = 0$$

But $$\lim_{b\to\infty} \int_{0}^{b} x\,dx = \infty$$

Thus while the integral from 0 to ∞ and the integral from $-\infty$ to 0 may be meaningless, the Symmetric Limit Ⓒ may exist

If equation Ⓑ is convergent, Then $I = I_p$. In applications the convergent of I is checked by using the usual tests of calculus and the integral is evaluated via Ⓒ

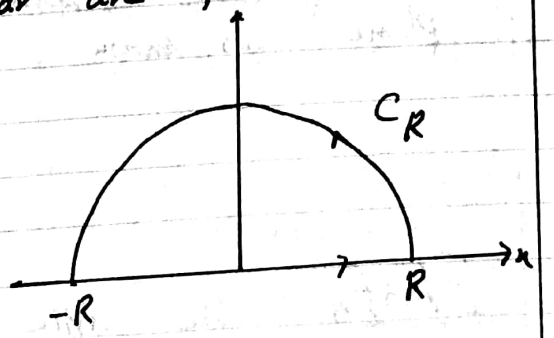
Some Important Lemmas To Evaluate The Contour Integral

The most difficult part of the determination of a contour integral is frequently the demonstration that the integral is zero on some part of the contour. The following Theorems are often used for this purpose.

Theorem # If $|f(z)| \leq \frac{M}{R^k}$ on $z = Re^{i\theta}$ i.e $|z| = R$, where $k > 1$ and $M > 0$ are constants, then prove that

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

where C_R is semi-circular arc of Radius R



Proof #

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f(z)|\,|dz|$$

$$\le \frac{M}{R^k}\cdot \pi R = \frac{\pi M}{R^{k-1}}$$

where $\int_{C_R} |dz|$ = length of the arc

Then

$$\lim_{R\to\infty}\left|\int_{C_R} f(z)\,dz\right| = 0 \quad \text{and so} \quad \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$$

Problem # Show that for $z = Re^{i\theta}$, $|f(z)| \le \frac{M}{R^k}$, $k>1$ if $f(z) = \frac{1}{1+z^4}$

Sol # If $z = Re^{i\theta}$,

$$|f(z)| = \left|\frac{1}{1+R^4 e^{4i\theta}}\right|$$

$$\le \frac{1}{|R^4 e^{4i\theta}| - 1} = \frac{1}{R^4-1} \le \frac{2}{R^4}$$

$$\frac{1}{R^4-1} = \frac{1+1}{R^4-1+1} = \frac{2}{R^4}$$

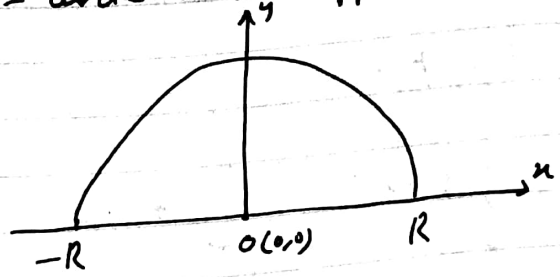
If R is large enough (say $R>2$, for example) so that $M=2$, $k=4$

We have made the use of inequality $|z_1+z_2| \ge |z_1| - |z_2|$

Theorem # Let $f(z) = \frac{N(z)}{D(z)}$ be a rational function such that degree of $D(z)$ exceeds the degree of $N(z)$ by at least two i.e if (degree of $N(z)$) $+2$ $\le$ (degree of D) (Note that it also means that $f(z)$ a zero of order greater than 1 at ∞). Then

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$$

Where C_R is a semi-circle in the upper half plane as shown



Proof# We write

$$f(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}$$

where $m \geq n+2$

$z = Re^{i\theta}$

$dz = Re^{i\theta} d\theta$

$|dz| = R\, d\theta$

$$\left| \oint_{C_R} f(z)\, dz \right| \leq \int_{C_R} |f(z)|\, |dz|$$

$$\leq \int_0^{\pi} (R\, d\theta) \cdot \frac{|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|}{|b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0|}$$

$$\leq \int_0^{\pi} (R\, d\theta) \; \frac{|a_n||z^n| + |a_{n-1}||z^{n-1}| + \cdots + |a_0|}{|b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0|}$$

$$\leq \int_0^{\pi} (R\, d\theta) \; \frac{|a_n|\,|z^n| + |a_{n-1}|\,|z^{n-1}| + \cdots + |a_0|}{|b_m|\,|z^m| - |b_{m-1}|\,|z^{m-1}| - \cdots + |b_1|\,|z| + |b_0|}$$

$$= \frac{\pi R \{ |a_n| R^n + |a_{n-1}| R^{n-1} + \cdots + |a_0| \}}{|b_m| R^m - |b_{m-1}| R^{m-1} - \cdots - |b_0|}$$

$\longrightarrow 0$ as $R \to \infty$ because $m \geq n+2$

Evaluation of $\int_{-\infty}^{+\infty} f(z)\, dz$

Theorem# Let $f(z)$ be a function such that

(i) $f(z)$ is analytic throughout the upper half plane except at certain points which are its poles

(ii) $f(z)$ has no poles on the real axis (i.e. no real poles)

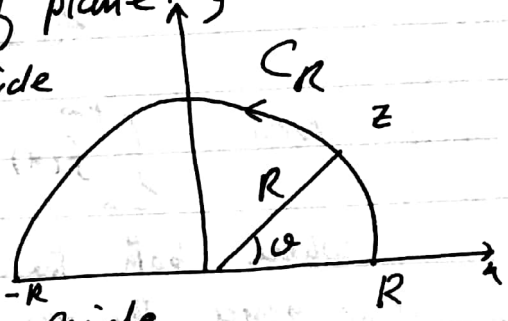
i.e if $R \to \infty$ (R radius of semi-circle), then it will

(iii) $z f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \le \arg \le \pi$

(iv) $\int_0^{\infty} f(u)\,du$ and $\int_{-\infty}^{0} f(u)\,du$ both converge, then

$$\int_{-\infty}^{+\infty} f(u)\,du = 2\pi i \sum R^{+}$$

where $\sum R^{+}$ is the sum of residues of $f(z)$ at its poles in the upper half plane.



Proof Choosing a semi-circle as contour let the origin be the centre of the semi-circle and R its radius in the upper half plane. Let semi-circle be represented by C_R and R be chosen large enough to include all the poles of $f(z)$.

If $f(z)$ has no poles on the boundry and we consider the closed contour C, then by residue theorem

$$\int_C f(z)\,dz = \int_{-R}^{R} f(u)\,du + \int_{C_R} f(z)\,dz$$

Since along x-axis $z = x$ $\therefore f(u) = f(z)$

Assuming that $z = Re^{i\theta}$ $dz = iRe^{i\theta}d\theta$.

$= iz\,d\theta$

we have

$$\int_{C_R} f(z)\,dz = i\int_0^{\pi} f(z)\,z\,d\theta$$

$= 0$

Since $\left|\int_0^{\pi} f(z)\,z\,d\theta\right| \le \int_0^{\pi} |f(z)|\,|z|\,d\theta$

≤ 0 as $z \to \infty$ $z f z \to 0$

$= 0$ as modulus cannot be -ve

Hence by Cauchy's Residue Theorem

$$2\pi i \sum R^+ = \int_C f(z)\,dz$$

$$= \int_{-R}^{R} f(x)\,dx + 0$$

$$= \int_{-\infty}^{+\infty} f(u)\,du \quad \text{as } R \to \infty$$

Now $$\int_{-\infty}^{+\infty} f(u)\,du = \int_0^{\infty} f(u)\,du + \int_{-\infty}^{0} f(u)\,du$$

where both the integrals on R.H.S individually converge and hence the result.

Remarks # The use of upper semi-circle is of course only a matter of convenience. One may as well use the lower semi-circle or any other suitable contour having the real interval $[-R, R]$ as its part.

Theorem # Let f be a rational function having no real poles such that there is R_1, such that for $|z| > R_1$, $|f(z)| \leq M/|z|^2$ for some $M > 0$. Then

$$\int_{-\infty}^{+\infty} f(u)\,du = 2\pi i \sum \text{Res}(f, p_j)$$ the sum being taken over all the poles p_j of f which fall in the upper half plane

Proof # Let $R > R_1$ be such that the semi-circle D_R contains all the poles of f in the upper half plane. Then by Residue Theorem

$$\int_{D_R} f\,dz = \int_{[-R,R]} f(z)\,dz + \int_{C_R} f(z)\,dz = 2\pi i \sum \text{Res}(f, p_j)$$

where β_j's are the poles in the upper half plane. The right side is independent of R while we get on the left side

$$\lim_{R\to\infty} \int_{[-R,R]} f(z)dz = \int_{-\infty}^{+\infty} f(x)dx \qquad \text{along } x\text{-axis } z=x$$

and

$$\left| \int_{C_R} f(z)dz \right| = \left| \int_0^{\pi} f(Re^{it}) Re^{it} dt \right|$$

$$\leq \int_0^{\pi} |f(Re^{it})| R\, dt$$

$$\leq \int_0^{\pi} \frac{MR}{R^2} dt = \frac{M}{R}\pi$$

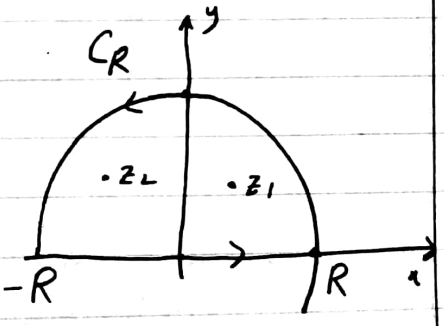
so that

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0 \quad \text{Hence}$$

$$\lim_{R\to\infty} \int_{D_R} f(z)dz = \int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum \text{Res}(f, \beta_j)$$

Example # Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^4}$ by Contour integration

Sol # This integral can be regarded as a line integral of $f(z) = \frac{1}{z^4+1}$ along the real axis. The path is not closed (unless one adjoins ∞), but we show that it acts like a closed path "enclosing" the upper half plane, so the integral along the path equals the sum of the residues in the upper half-plane.



To establish this, we consider the integral of $f(z)$ on the semi-circular path C_R as shown

The poles are roots of $z^4 = -1 = \cos\pi + i\sin\pi = e^{i\pi}$

$$z_k = 1^{1/4}\left[\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)\right]$$

$$z_0 = \cos\pi/4 + i\sin\pi/4 = e^{i\pi/4}$$

$$z_1 = \cos 3\pi/4 + i\sin 3\pi/4 = e^{i3\pi/4}$$

$$z_2 = \cos 5\pi/4 + i\sin 5\pi/4 = e^{i5\pi/4}$$

$$z_3 = \cos 7\pi/4 + i\sin 7\pi/4 = e^{i7\pi/4}$$

The poles $e^{i\pi/4}$, $e^{i3\pi/4}$ lie inside the Contour

Residue at $z = e^{i\pi/4} = \lim_{z\to e^{i\pi/4}} \frac{(z - e^{i\pi/4})}{z^4+1}$. (1/2)

$$= \lim_{z\to e^{i\pi/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4e^{i3\pi/4}}$$

$$= \frac{1}{4}e^{-i3\pi/4}$$

$$= \frac{1}{4}\left(\cos 3\pi/4 - i\sin 3\pi/4\right)$$

$$= \frac{1}{4}\left[\cos(\pi - \pi/4) - i\sin(\pi - \pi/4)\right]$$

$$= \frac{1}{4}\left[-\cos\pi/4 - i\sin\pi/4\right]$$

$$= \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right]$$

Residue at $z = e^{i3\pi/4} = \lim_{z\to i3\pi/4} \frac{z - e^{i3\pi/4}}{z^4+1}$ (1/2)

$$= \lim_{z\to i3\pi/4} \frac{1}{4z^3} = \frac{1}{4e^{i9\pi/4}}$$

$$= \frac{1}{4}e^{-i9\pi/4}$$

$$= \frac{1}{4}\left[\cos 9\pi/4 - i\sin 9\pi/4\right]$$

$$= \frac{1}{4}\left[\cos(2\pi + \pi/4) - i\sin(2\pi + \pi/4)\right]$$

$$= \frac{1}{4}\left[\cos \pi/4 - i\sin \pi/4\right]$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{2}} - i\cdot\frac{1}{\sqrt{2}}\right]$$

Sum of Residues $= \frac{1}{4}\left(-\frac{2}{\sqrt{2}}i\right) = -\frac{\sqrt{2}}{4}i$

OR

Residue of $\frac{P(z)}{Q(z)}$ at pole $= \left[\frac{P(z)}{Q'(z)}\right]_{\text{at pole}}$

Residue at $z_1 = e^{i\cdot\pi/4} = \frac{1}{4z_1^3}$

Residue at $z_2 = e^{i\,3\pi/4} = \frac{1}{4z_2^3}$

Sum of the residues $= \frac{1}{4}\left[\frac{1}{z_1^3} + \frac{1}{z_2^3}\right]$

$$= \frac{1}{4}\left[\frac{z_2^3 + z_1^3}{z_1^3 z_2^3}\right]$$

$$= \frac{1}{4}\left[\frac{z_2^2 \cdot z_2 + z_1^2 \cdot z_1}{(z_1 z_2)^3}\right]$$

$$= \frac{1}{4}\left[\frac{e^{i\,3\pi/2}\cdot z_2 + e^{i\pi/2}\cdot z_1}{(e^{i\pi})^3}\right]$$

$$= -\frac{\sqrt{2}}{4}i$$

Let C be Contour including semi-circle C_R together with real axis from $-R$ to R.

As R increases, the integral on C cannot change, since it always equals the sum of residues times $2\pi i$. Hence

$$\oint_C f(z)\,dz = 2\pi i\,(\text{Sum of residues})$$

$$\int_{-R}^{R} \frac{dx}{x^4+1} + \int_{C_R} \frac{1}{z+1}\,dz = 2\pi i\left(-\frac{\sqrt{2}}{4}i\right) = \frac{\sqrt{2}\,\pi}{2}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+1}\,dx + \lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1}\,dz = \frac{\pi\sqrt{2}}{2}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{x^4+1}\,dx + \lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1}\,dz = \frac{\pi\sqrt{2}}{2}$$

Now

$$\left| \int_{C_R} \frac{1}{z^4+1}\,dz \right| \le \int_{C_R} \left|\frac{1}{z^4+1}\right| |dz|$$

$$\le \int_0^{\pi} \frac{1}{|z|^4-1}\,|dz|$$

$z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$, $|dz| = R\,d\theta$

$$= \frac{\pi R}{R^4-1} \to 0 \quad \text{as } R\to\infty$$

$$\Rightarrow \int_{C_R} \frac{1}{z^4+1}\,dz = 0$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}\,\pi}{2}$$

Note #

$$\int_{-\infty}^{+\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}\,\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{1}{x^4+1}\,dx + \int_{0}^{+\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}\,\pi}{2}$$

$$\int_{0}^{\infty} \frac{1}{x^4+1}\,dx + \int_{0}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}\,\pi}{2}$$

$$2\int_{0}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{2}\pi$$

$$\int_{0}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{4}\pi$$

$$-\int_{\infty}^{-\infty} \frac{1}{x^4+1}\,dx$$

$x = -t$, $dx = -dt$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

if $f(x)$ is even

Example # Prove that $\int_0^\infty \frac{z^6}{(a^4+z^4)^2}\,dz$

Sol # $f(z) = \frac{z^6}{(a^4+z^4)^2}$

$z f(z) \to 0$ as $z \to \infty$ which satisfies the condition for evaluation of an infinite interval

Poles of $f(z)$ are given by

$z^4 = -a^4$

$= a^4(\cos\pi + i\sin\pi)$

$= a^4 e^{i\pi}$

$z_n = a\, e^{\frac{(2n\pi+\pi)i}{4}}$ $\quad n = 0, 1, 2, 3$

$= a\, e^{\frac{(2n+1)\pi i}{4}}$

$z_1 = -a^4$, $|z_1| = a^4$, Case = -1, Sin a > 0

Thus the poles are $ae^{\frac{\pi i}{4}}$, $ae^{\frac{3\pi i}{4}}$, $ae^{\frac{5\pi i}{4}}$, $ae^{\frac{7\pi i}{4}}$ of which the 1st two only lie in the upper half plane (Co-efficient of i being +ve). which is chosen as Contour Consisting of Large semi-circle C_R along with real axis from $-R$ to R.

Applying Residue Theorem

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \,(\text{Sum of Residues})$$

where C_R represents the upper half plane

When $R \to \infty$ $\quad \int_{C_R} f(z)\,dz \to 0$

Hence when $R \to \infty$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \,(\text{Sum of Residues})$$

To find the residue at $z = ae^{\pi i/4} = z_1$ let us put $z = ae^{\pi i/4} + t = z_1 + t$, t be is small in $f(z)$

$$f(z_1+t) = \frac{(z_1+t)^6}{[a^4+(z_1+t)^4]^2} = \frac{z_1^6 + 6z_1^5 t + \cdots}{[a^4 + z_1^4 + 4z_1^3 t + 6z_1^2 t^2 + 4z_1 t^3 + t^4]^2}$$

$$= \frac{z_1^6 + 6z_1^5 t + \cdots}{16 z_1^6 t^8 \left[1 + \frac{6t}{4z_1} + \cdots\right]^2}$$

$$= \frac{z_1^6 + 6z_1^5 t + \cdots}{16 z_1^6 t^8} \left[1 + \frac{6t}{4z_1} + \cdots\right]^{-2}$$

$$= \frac{z_1^6 + 6z_1^5 t}{16 z_1^6 t^8} \left[1 - \frac{12t}{4z_1} + \cdots\right]$$

Residue at $z = z_1$ = Co-efficient of $\frac{1}{t}$ in $f(t+z_1)$

$$= \frac{6}{16 z_1} - \frac{3}{16 z_1} = \frac{3}{16 z_1}$$

Similarly, residue at $z = z_2 = \frac{3}{16 z_2}$

Hence

$$\int_{-\infty}^{+\infty} \frac{x^6 dx}{(a^4 + x^4)^2} = 2\pi i \cdot \frac{3}{16}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)$$

$$= \frac{3\pi i}{8}\left(\frac{1}{a e^{i\pi/4}} + \frac{1}{a e^{3\pi i/4}}\right)$$

$$= \frac{3\pi i}{8a}\left(e^{-i\pi/4} + e^{-3\pi i/4}\right)$$

$$= \frac{3\pi i}{8a}\left(e^{-i\pi/4} - e^{i\pi/4}\right)$$

$$\therefore e^{-3\pi i/4} = e^{-\pi i} \cdot e^{\pi i/4} = -e^{\pi i/4}$$

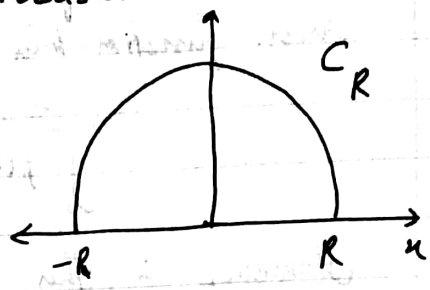
$$= -\frac{3\pi i}{8a}\left(2i \sin \pi/4\right)$$

$$= \frac{3\pi}{8a} \cdot \frac{1}{\sqrt{2}} = \frac{3\sqrt{2}\,\pi}{16a}$$

Example # Show that $\int_{-\infty}^{+\infty} \frac{x^2-1}{x^4+5x^2+4}\,dx = \pi/6$

Sol The degree of the denominator is two units higher than that of numerator. So

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2-1}{z^4+5z^2+4}\,dz = 0$$



Now

$$f(z) = \frac{z^2-1}{z^4+5z^2+4} = \frac{z^2-1}{(z^2+4)(z^2+1)} = \frac{p(z)}{q(z)}$$

has simple at $2i$, i in the upper half plane (and at $-2i$ & $-i$ in the lower half plane)

$$\frac{p(z)}{q'(z)} = \frac{z^2-1}{3z^3+10z}$$

$$\text{Res}(f, 2i) = \left[\frac{z^2-1}{3z^3+10z}\right]_{z=2i} = \frac{5}{12i}$$

$$\text{Res}(f\ i) = \left[\frac{z^2-1}{3z^3+10z}\right]_{z=i} = \frac{-2}{6i}$$

$$\int_{-\infty}^{+\infty} \frac{x^2-1}{x^4+5x^2+4}\,dx = 2\pi i\ (\text{Sum of Residues})$$

$$= 2\pi i\left(\frac{5}{12i} - \frac{2}{6i}\right)$$

$$= 2\pi i\left(\frac{5-4}{12i}\right)$$

$$= 2\pi i\left(\frac{1}{12i}\right) = \pi/6$$

Example # Evaluate $I = \int_0^{\infty} \frac{dx}{x^3+a^3}$ $a>0$

(Contour Used not Semi-Circle)

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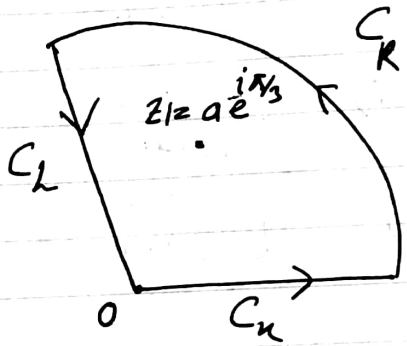
Sol # Because we have an integral on $(0, \infty)$, we cannot immediately use a contour like semi-circle in the upper or lower half plane.

If the integral was $\int_0^\infty f(u)\,du$ where $f(u)$ is even function. Then

$$\int_0^\infty f(u)\,du = \frac{1}{2}\int_{-\infty}^{+\infty} f(u)\,du.$$

However in this case the integrand is not even and for $u<0$ has a singularity. Nevertheless there is a symmetry that can be employed. namely $\left(x e^{2\pi i/3}\right)^3 = x^3$. This suggest using the contour like shown in fig below



where C_R is the sector $Re^{i\theta} = z \quad 0 \le \theta \le \frac{2\pi}{3}$
closed path $C = C_x + C_R + C_L$ and

$$\oint_C \frac{dz}{z^3+a^3} = \left(\int_{C_L} + \int_{C_x} + \int_{C_R}\right) \frac{dz}{z^3+a^3}$$

$$= 2\pi i \sum_j \text{Res}\left(\frac{1}{z^3+a^3} : z_j\right) \to (A)$$

The only pole inside satisfies $z^3 = -a^3 = a^3 e^{i\pi}$
and is given by $z_1 = a e^{i\pi/3}$

$$\text{Res}\left(\frac{1}{z^3+a^3} : z_1\right) = \left(\frac{1}{3z^2}\right)_{z_1} = \frac{1}{3a^2 e^{2\pi i/3}} = \frac{1}{3a^2} e^{-2\pi i/3}$$

Integral on C_R

$$\left|\int_{C_R} \frac{dz}{z^3+a^3}\right| \leq \int_{C_R} \left|\frac{1}{z^3+a^3}\right| |dz|$$

$$\leq \int_{C_R} \frac{1}{|z|^3 - a^3} |dz|$$

$$\leq \frac{1}{R^3 - a^3} \int_{C_R} |dz| =$$

$$\leq \frac{1}{R^3 - a^3} \cdot \frac{2\pi}{3} R \to 0 \quad (R \to \infty)$$

$$\int_{C_R} |dz| = \frac{2\pi}{3} R. \quad \to (1)$$

Integral on C_L

The integral on C_L is evaluated by making substitution $z = re^{2\pi i/3}$ (where orientation is taken into account)

$$\int_{C_L} \frac{dz}{z^3+a^3} = \int_{r=R}^{0} \frac{e^{2\pi i/3} dr}{r^3 e^{2\pi i} + a^3} \qquad z = re^{2\pi i/3}, \; dz = e^{2\pi i/3} dr$$

$$= \int_{r=R}^{0} \frac{e^{2\pi i/3} dr}{r^3 + a^3}$$

$$= -\int_{0}^{R} \frac{e^{2\pi i/3} dr}{r^3+a^3} = -e^{2\pi i/3} \int_0^R \frac{dr}{r^3+a^3} \quad \to (2)$$

Integral on C_H

Taking into account the contribution from C_H $(0 \leq z = x \leq R)$

$$\int_{C_x} \frac{dz}{z^3+a^3} = \int_0^R \frac{dx}{x^3+a^3} \quad \text{42}$$

$$= \int_0^R \frac{dr}{r^3+a^3} \rightarrow (3)$$

$$\int_a^b f(x)\,dx \quad 2\int_a^b f(z)\,dz$$

using ① . ② & ③ in (A)

$$-e^{2\pi i/3}\int_0^R \frac{dr}{r^3+a^3} + \int_0^R \frac{dr}{r^3+a^3} = 2\pi i \cdot \frac{e^{-2\pi i/3}}{3a^2}$$

$$\left(1-e^{2\pi i/3}\right)\int_0^R \frac{dr}{r^3+a^3} = \frac{2\pi i}{3a^2}\, e^{-2\pi i/3}$$

$$\int_0^R \frac{dr}{r^3+a^3} = \frac{2\pi i}{3a^2} \cdot \frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}}$$

$$= \frac{2\pi}{3a^2}\left(\frac{2i}{e^{-i\pi/3}+e^{i\pi/3}}\right)e^{-i\pi}$$

$$= \frac{2\pi}{3a^2}\cdot\frac{1}{\sin\pi/3} = \frac{2\pi}{3\sqrt{3}\,a^2}$$

$$\int_0^R \frac{dx}{x^3+a^3} = \frac{2\pi}{3\sqrt{3}\,a^2}$$

$$\lim_{R\to\infty} \int_0^R \frac{dx}{x^3+a^3} = \frac{2\pi}{3\sqrt{3}\,a^2}$$

$$\int_0^\infty \frac{dx}{x^3+a^3} = \frac{2\pi}{3\sqrt{3}\,a^2}.$$

Example # Prove that $\int_0^\infty \frac{\log(1+x^2)}{1+x^2} = \pi \log 2$

Sol Take $f(z) = \frac{\log(1+z^2)}{1+z^2}$

and choose the contour C consisting of upper half C_R of circle $|z|=R$ (R large) and real axis from $-R$ to R

Poles of $f(z)$ are roots of $z^2+1=0$

$$z = \pm i$$

The only pole lying within C is i

$$R(f,i) = i \lim_{z\to i} (z-i) \frac{\log(z+1)}{1+z^2}$$

$$= \lim_{z\to i} (z-i) \frac{\log(z+1)}{(z-i)(z+i)}$$

$$= \lim_{z\to i} \left(\frac{\log 2i}{2i}\right)$$

$$= \frac{\log 2 + i\pi/2}{2i} \quad \log i = \log e^{i\pi/2}.$$

Hence by Cauchy Residue Theorem

$$\int_{-R}^{R} f(u)\,du + \int_{C_R} f(z)\,dz = 2\pi i \,(\text{Sum of Residues})$$

$$= 2\pi i \times \frac{1}{2i}\left(\log 2 + i\pi/2\right)$$

$$= \pi\left[\log 2 + i\pi/2\right]$$

where

$$\left| \int_{C_R} \frac{\log(z+i)}{z^2+1}\,dz \right| \le \int_{C_R} \frac{|\log(z+i)|}{|z^2+1|}\,|dz|$$

$$\le \int_{C_R} \frac{|\log|z+i| + i\,\mathrm{Arg}(z+i)|}{|z|^2-1}\,|dz|$$

$$\leq \int_{C_R} \frac{|\log(|z|+1) + 2\pi i|}{|z|^2-1}\,|dz| \qquad \because \arg(z+i) \leq 2\pi$$
$$|i| = 1$$

$$\leq \frac{\{\log((R+1)+2\pi)\}\pi R}{R^2-1}$$

$$\leq \left[\frac{\log(R+1)}{R+1} + \frac{2\pi}{R+1}\right]\frac{\pi R}{R-1} \to 0 \quad \text{as } R\to\infty$$

Hence we have $R\to\infty$

$$\int_{-\infty}^{+\infty} f(u)\,du = \int_{-\infty}^{+\infty} \frac{\log(x+i)}{x^2+1}\,dx = \pi\left(\log 2 + i\frac{\pi}{2}\right)$$

$$= \int_{-\infty}^{+\infty} \frac{\frac{1}{2}\log(1+x^2) + i\tan^{-1}\left(\frac{1}{x}\right)}{x^2+1}\,dx = \pi\left(\log 2 + i\frac{\pi}{2}\right)$$

Comparing real and imaginary parts, we get

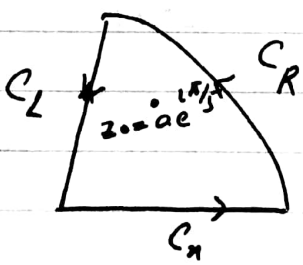
$$\int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(1+x^2)}{1+x^2} = \pi\log 2$$

$$\therefore\ 2\int_{0}^{\infty} \frac{1}{2}\,\frac{\log(1+x^2)}{1+x^2} = \pi\log 2$$

$$\int_{0}^{\infty} \frac{\log(1+x^2)}{1+x^2} = \pi\log 2$$

Example # prove that $\int_0^\infty \frac{dx}{x^5+a^5} = \frac{\pi}{5a^4\sin\pi/5}$ for $a>0$

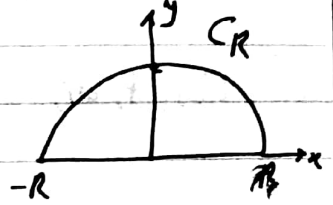
Sol Here $\left(xe^{\frac{2\pi i}{5}}\right)^5 = x^5$. This suggests the Contour like shown in fig



Example # Prove that $\int_{-\infty}^{+\infty} \frac{dx}{x^2+x+1} = 2\pi/\sqrt{3}$

Sol # Here the denominator has degree two units greater than the degree of numerator. Hence

$$\int_{C_R} \frac{1}{z^2+z+1}\,dz = 0$$



$f(z) = \frac{1}{z^2+z+1}$ has two poles at $z_1 = \frac{-1+\sqrt{3}i}{2}$

$z_2 = \frac{-1-\sqrt{3}i}{2}$ of which only z_1 is in upper half plane and neither on the real axis

$$I = \int_{-\infty}^{+\infty} \frac{dx}{x^2+x+1} = 2\pi i\, Res(f, z_1)$$

$$= 2\pi i \left[\frac{1}{2z+1}\right]_{z_1}$$

$$= 2\pi i \left[\frac{1}{-1+\sqrt{3}i+1}\right]$$

$$= \frac{2\pi}{\sqrt{3}}$$

Example # Prove that $I = \int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \pi/4a^3$. $a>0$

Sol The integrand is even

$$\Rightarrow I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2}$$

The function $f(z) = \frac{1}{(z^2+a^2)^2}$ has double poles at $\pm ai$. only ai lies in the upper half plane. Also the degree of Denomi

exceeds that of the numerator by 4.

Hence

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\cdot 2\pi i \cdot \text{Res}(f, ai)$$

$$= \pi i \, \text{Res}(f, ai)$$

$$\text{Res}(f, ai) = \frac{1}{1!}\lim_{z\to ai}\frac{d}{dz}(z+ai)^{-2}$$

$$= \lim_{z\to ai}\frac{-2}{(z+ai)^3} = \frac{1}{4a^3 i} \quad \text{so that}$$

$$I = \pi i \times \frac{1}{4a^3 i} = \frac{\pi}{4a^3}$$

Example # prove $\int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$

Sol Let $f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$

Poles of $f(z)$ are solution of $z^2+1=0$ & $z^2+2z+2=0$

$z^2+1=0 \Rightarrow z = \pm i$

$$z^2+2z+2=0$$

$$z = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2}$$

$$= \frac{-2\pm 2i}{2} = -1\pm i$$

The poles in the upper half plane are $z_1 = i$, $z_2 = -1+i$

Residue at $z = i$ $= \lim_{z\to i}\frac{d}{dz}\left\{(z-i)\frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\}$

$$= \lim_{z\to i}\frac{d}{dz}\left\{\frac{z^2}{(z+i)^2(z-i)(z^2+2z+2)}\right\}$$

$$= \lim_{z\to i}\left\{2z(z+i)^{-2}(z-i)(z^2+2z+2)^{-1} + 2z^2(z+i)^{-3}(z-i)^{-1}(z^2+2z+2)^{-1}\right.$$

$$\left. - z^2(z+i)^{-2}(z-i)^{-2}(z^2+2z+2)^{-1} - z^2(z+i)^{-2}(z-i)^{-1}(z^2+2z+2)^{-2}(2z+2)\right]$$

$$= \frac{9i-12}{100}$$

Residue at $z = -1+i = \lim_{z\to -1+i}(z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$

$$\int_{-\infty}^{+\infty}\frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = 2\pi i\left\{\frac{9i-12}{100} + \frac{3-4i}{25}\right\}$$

$$= \frac{7\pi}{50}$$

Where the closed contour C is the sector $Re^{i\theta}$

$0 \le \theta \le 2\pi/5$

$$\oint_C \frac{dz}{z^5+a^5} = \left(\int_{C_L} + \int_{C_R} + \int_{C_R}\right) \frac{dz}{z^5+a^5}$$

$$= 2\pi i \sum_j \text{Res}\left(\frac{1}{z^5+a^5} : z_j\right) \longrightarrow (A)$$

The poles are roots of $z^5+a^5=0$

Now

$$z^5+a^5 = 0$$

$$\Rightarrow z^5 = -1\,a^5 = a^5(\cos\pi + i\sin\pi)$$

$$z_k = a\left[\cos\left(\frac{2k\pi+\pi}{5}\right) + i\sin\left(\frac{2\pi k+\pi}{5}\right)\right]$$

$k = 0, 1, 2, 3, 4$

$z_0 = a\left[\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right] = ae^{i\pi/5}$

$z_1 = a\left[\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right] = ae^{i3\pi/5}$

$z_2 = a\left[\cos\left(\frac{5\pi}{5}\right) + i\sin\left(\frac{5\pi}{5}\right)\right] = -a$

$z_3 = a\left[\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right] = ae^{i7\pi/5}$

$z_4 = ae^{i9\pi/5}$

only $z_0 = ae^{i\pi/5}$ lies in the contour

$$\text{Res}\left(\frac{1}{z^5+a^5} : z_0\right) = \left(\frac{1}{5z^4}\right)_{z_0} = \frac{1}{5a^4e^{4\pi/5}i}$$

$$= \frac{1}{5a^4}e^{-4\pi/5 i} \longrightarrow (1)$$

Integral on C_R

$$\left|\int_{C_R} \frac{dz}{z^5+a^5}\right| \le \int_{C_R} \left|\frac{1}{z^5+a^5}\right| |dz|$$

$$\le \int_{C_R} \frac{1}{|z|^5 - a^5} |dz|$$

$$\leq \frac{1}{R^5 - a^5} \overset{48}{\int_{C_R}} |dz| = \frac{1}{R^5 - a^5} \frac{2\pi}{5} R \to 0 \quad \text{as } R \to \infty$$

$$\Rightarrow \int_{C_R} \frac{dz}{z^5 + a^5} = 0 \longrightarrow (2)$$

Integral on C_2

The integral on C_2 is evaluated by making substitution $z = re^{\frac{2\pi}{5}i}$ (where orientation is taken into account)

$$\int_{C_2} \frac{dz}{z^5 + a^5} = \int_{r=R}^{0} \frac{e^{2\pi/5\, i}\, dr}{r^5 e^{2\pi i} + a^5}$$

$$= \int_{r=R}^{0} \frac{e^{\frac{2\pi}{5}i}\, dr}{r^5 + a^5}$$

$$= -\int_{0}^{r=R} \frac{e^{2\pi/5\, i}}{r^5 + a^5} \longrightarrow (3)$$

Integral on C_x

Taking into account the contribution from C_x $(0 \leq z = x \leq R)$

$$\int_{C_x} \frac{dz}{z^5 + a^5} = \int_{0}^{R} \frac{dx}{x^5 + a^5}$$

$$= \int_{0}^{R} \frac{dr}{r^5 + a^5} \qquad \because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz$$

$$\longrightarrow (4)$$

Using (1), (2), (3) & (4) in (A)

$$-e^{\frac{2\pi}{5}i}\int_0^R \frac{dr}{r^5+a^5} + \int_0^R \frac{dr}{r^5+a^5} = 2\pi i \cdot \frac{1}{5a^4} e^{-\frac{4\pi}{5}i}$$

$$\left(1-e^{\frac{2\pi}{5}i}\right)\int_0^R \frac{dr}{r^5+a^5} = \frac{2\pi i}{5a^4}\cdot e^{-\frac{2\pi}{5}i}$$

$$\Rightarrow \int_0^R \frac{dr}{r^5+a^5} = \frac{\pi}{5a^4}\left(\frac{2i}{1-e^{\frac{2\pi i}{5}}}\right)\cdot e^{-\frac{2\pi i}{5}}$$

$$= \frac{\pi}{5a^4}\left(\frac{2i}{e^{-\frac{\pi i}{5}} - e^{\frac{\pi i}{5}}}\right)$$

$$\int_0^R \frac{dx}{x^5+a^5} = \frac{\pi}{5a^4}\frac{1}{\sin\pi/5}$$

when $R \to \infty$, we have.

$$\int_0^\infty \frac{dx}{x^5+a^5} = \frac{\pi}{5a^4 \sin\pi/5} \quad \text{(proved)}$$

<u>Example</u># prove that $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$

Sol $f(z) = \frac{1}{(z^2+1)^3}$

$z f(z) \to 0$ as $|z| \to \infty$

The poles of $f(z)$ are at $z = \pm i$ of order 3.
The pole $z = i$ lies in the upper half plane

$$R(f,i) = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-i)^3 \cdot \frac{1}{(z+i)^3(z-i)^3}\right]_{z=i}$$

$$= \frac{1}{2}\frac{d^2}{dz^2}(z+i)^{-3} = \frac{1}{2}(-3)(-4)\cdot\frac{1}{(2i)^5} = \frac{6}{32i}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^3} = 2\pi i \times \frac{6}{32i} = \frac{3\pi}{8}$$

Example # prove that $\int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = \frac{\pi}{8a^3}$ provided $R(a)$ is +ve

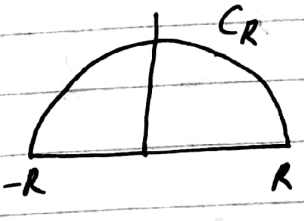
Sol # $I = \int_{-\infty}^{+\infty} \frac{x^2}{(x^2+a^2)^3}\,dx$

what is the value of integral if $R(a)$ is -ve

$$f(z) = \frac{z^2}{(z^2+a^2)^3}$$

∴ Degree of the denominator exceeds by 4 than that of the numerator.

Therefore $\int_{C_R} f(z)\,dz = 0$



poles of $f(z)$ are at $z = \pm ai$ of order 3.

pole $z = ai$ lies in the upper half plane.

$$R(f, ai) = \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z^2(z-ai)^3}{(z-ai)^3(z+ai)^3}\right]_{z=ai}$$

$$= \frac{1}{2!}\frac{d}{dz}\left[\frac{(z+ai)^3 \cdot 2z - 3z^2(z+ai)^2}{(z+ai)^6}\right]_{z=ai}$$

$$= \frac{1}{2}\frac{d}{dz}\left[\frac{2aiz - z^2}{(z+ai)^4}\right]_{z=ai}$$

$$= \frac{1}{2}\left[\frac{4a^2 \cdot 8a^3(-i)}{(2a)^8}\right] = -\frac{i}{16a^3} = \frac{1}{16a^3 i}$$

$$\int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = 2\pi i \times \frac{1}{16a^3 i} = \frac{\pi}{8a^3} \rightarrow (A)$$

If $R(a)$ is -ve the pole which lies in the upper half plane is $z = -ai$

putting $-a$ for a in (A), we get

$$\int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = -\frac{\pi}{8a^3}$$

Example # Evaluate $\int_0^\infty \frac{dx}{x^4+a^4}$ $(a>0)$

Sol Integrand is even.

$$\therefore \int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{x^4+a^4}$$

$$f(z) = \frac{1}{z^4+a^4}$$

The poles of $f(z)$ are at

$$z^4+a^4 = 0$$

$$z^4 = -a^4 = a^4(\cos\pi + i\sin\pi) = a^4 e^{i\pi}$$

$$z_k = a\, e^{\frac{(2k\pi+\pi)i}{4}} \qquad k=0,1,2,3$$

$$z_0 = a e^{i\pi/4}$$

$$z_1 = a e^{3\pi i/4} \qquad z_2 = a e^{5\pi i/4}$$

$$z_3 = a e^{7\pi i/4}$$

The only poles which lie in the upper half plane are $z_1 = a e^{\pi i/4}$ $z_2 = a e^{3\pi i/4}$

Let z_2 α be one of these poles

$$\alpha^4 + a^4 = 0 \Rightarrow \alpha^4 = -a^4$$

$$R_1(f,\alpha) = \lim_{z\to\alpha}\left[\frac{z-\alpha}{z^4+a^4}\right] \quad (0/0)$$

$$= \lim_{z\to\alpha} \frac{1}{4z^3} = \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4}$$

$$= \frac{\alpha}{-4a^4} \qquad \because \alpha^4 = -a^4$$

Sum of the residues $= -\frac{1}{4a^4}[\alpha+\beta]$

$$= -\frac{1}{4a^4}\left[a e^{\pi i/4} + a e^{3\pi i/4}\right]$$

$$= -\frac{\alpha}{4a^4}\left[\frac{e^{\pi i/4} + e^{+i\pi}\cdot e^{-\frac{\pi i}{4}}}{2i}\right]\times 2i$$

$$= -\frac{1}{4a^3}\left[\frac{e^{\pi i/4} - e^{-\pi i/4}}{2i}\right]\times 2i$$

$$= -\frac{2i}{4a^3}\sin\pi/4 = -\frac{2i}{4a^3}\cdot\frac{1}{\sqrt{2}} = -\frac{i}{2\sqrt{2}a^3}$$

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2}\cdot 2\pi i \times \frac{-i}{2\sqrt{2}a^3} = \frac{\pi}{2\sqrt{2}a^3}$$

Example # Evaluate $\int_0^\infty \frac{x^2\,dx}{x^6+1}$

Sol $f(z) = \frac{z^2}{z^6+1}$

The degree of the denominator is four units greater than the degree of numerator.

$$\Rightarrow \int_{C_R} f(z)\,dz = 0$$

poles are roots of

$$z^6 = -1 = e^{i\pi}$$

$$z_k = e^{\frac{(2k\pi+\pi)}{6}i} \qquad k = 0, 1, 2, 3, 4, 5$$

$$z_0 = e^{\pi i/6}$$

$$z_1 = e^{\pi i/2}$$

$$z_2 = e^{5\pi i/6}$$

$$z_3 = e^{7\pi i/6}$$

$$z_4 = e^{3\pi i/2} \qquad z_5 = e^{11\pi i/6}$$

The only poles which lie at the upper half plane are

$$z_0 = e^{\pi i/6} \qquad z_1 = e^{\pi i/2} \qquad z_2 = e^{5\pi i/6}$$

Let β be any of these poles.

Then $\beta^6 = -1$

$$R(f,\beta) = \lim_{z\to\beta} \frac{z^2(z-\beta)}{z^6+1} \qquad (0/0)$$

$$= \lim_{z\to\beta} \frac{3z^2 - 2z\beta}{6z^5} = \frac{\beta^2}{6\beta^5} = \frac{\beta^3}{6\beta^6}$$

$$= -\beta^3/6$$

$$\sum R^+ = -\frac{1}{6}\left[e^{3(\pi i/6)} + e^{3\pi i/2} + e^{\frac{5\pi i}{2}}\right]$$

$$= -\frac{1}{6}\left[i - i + i\right] = i/6$$

$$\int_0^{\infty} \frac{x^2 dx}{x^6+1} \overset{53}{=} \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2 dx}{x^6+1}$$

$$= \frac{1}{2} \times 2\pi i\,(-i/6) = \pi/6$$

Example # Evaluate $\int_{-\infty}^{+\infty} \frac{x^4 dx}{(a+bx^2)^4}$

Sol # $f(z) = \frac{z^4}{(a+bz^2)^4}$

Degree of the denominator is four unit greater than that of numerator.

The poles of $f(z)$ are roots of

$$a+bz^2 = 0 \Rightarrow z = \pm\sqrt{\frac{a}{b}}\, i$$

The only pole in the upper half plane is $\sqrt{\frac{a}{b}}\, i$

$$R\left(f, \sqrt{\tfrac{a}{b}}\, i\right) = \left[\frac{z-\sqrt{\frac{a}{b}}\, i}{(z-\sqrt{\frac{a}{b}}\, i)(z+\sqrt{\frac{a}{b}}\, i)}\right]_{z=\sqrt{\frac{a}{b}}\, i}$$

$$= \frac{1}{2i\sqrt{\frac{a}{b}}}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{a+bx^2} = \frac{1}{b} \times 2\pi i \times \frac{1}{2i \times \sqrt{a/b}}$$

$$= \frac{\pi}{\sqrt{a}\sqrt{b}}$$

Diff w.r.t. b

$$\int_{-\infty}^{+\infty} \frac{-bx^4\, dx}{(a+bx^2)^{+2}} = \frac{\pi}{\sqrt{a}} \cdot -\frac{1}{2} \cdot b^{-3/2}$$

$$\int_{-\infty}^{+\infty} \frac{x^2\, dx}{(a+bx^2)^2} = \frac{\pi}{\sqrt{a}} \cdot \frac{1}{2} b^{-3/2}$$

Diff. w.r.t. b

$$\int_{-\infty}^{+\infty} \frac{-2x^4}{(a+bx^2)^3} = \frac{\pi}{\sqrt{a}} \cdot \frac{1}{2}(-3/2)\, b^{-5/2}$$

$$\int_{-\infty}^{+\infty} \frac{x^4\,dx}{(a+bx^2)^3} = \frac{\pi}{\sqrt{a}}\,\frac{3}{8}\,b^{-5/2}$$

Diff w.r.t a

$$\int_{-\infty}^{+\infty} \frac{-3\,x^4\,dx}{(a+bx^2)^4} = \pi\left(-\tfrac{1}{2}\right)\cdot\frac{1}{a^{3/2}}\cdot\frac{1}{b^{5/2}}\cdot\frac{3}{8}$$

$$\int_{-\infty}^{+\infty} \frac{x^4\,dx}{(a+bx^2)^4} = \frac{\pi}{16\,a^{3/2}\,b^{5/2}} \quad \text{(proved)}$$

Real Integrals: $\int_{-\infty}^{+\infty} \cos ax\,dx$ & $\int_{-\infty}^{+\infty} \sin ax\,dx$, $a>0$

OR

Fourier Integrals

Suppose f is a rational function without real pole and with degree of the denominator is at least two unit greater than the degree of numerator. Let $|f(x)| \le M/|x|^2$ for x sufficiently large. Let $g = e^{iaz} f(z)$ for some real $a>0$. Then g has same pole as f and for any $z = x+iy$ in the upper half plane

$$|e^{iaz}| = |e^{iax}\cdot e^{-ay}| \le e^{-ay} \qquad |e^{iay}| \le 1$$
$$\le 1 \qquad \because y>0$$

$$|g(z)| \le |f(z)| \le M/|z|^2 \quad \text{for } z \text{ sufficiently large.}$$

Hence along the upper semi-circle C_R

$$\left|\int_{C_R} f(z)\,e^{iaz}\,dz\right| \le \int_0^R |f(z)|\,|dz| \to 0 \quad \text{as } R\to\infty$$

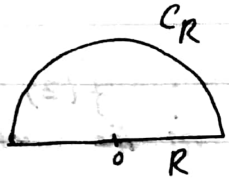
$$\int_{-\infty}^{+\infty} e^{iax} f(x)\,dx = 2\pi i \sum \text{Residues of } g(z) \text{ in the upper half plane.}$$

$$= \int_{-\infty}^{+\infty} \cos ax \cdot f(x)\,dx + i\int_{-\infty}^{+\infty} \sin ax\, f(x)\,dx$$

Equating the real and imaginary parts we get the values of the integral

Note # We note that in other applications one might need to consider integrals $\int_{C_R} e^{-az} f(z)\,dz$ where C_R is a semi-circle in the right half plane, and/or $\int_{C_L} e^{az} f(z)\,dz$ where C_L is a semi-circle in the left half plane. The methods to show such integrals are zero as $R \to \infty$ are similar.

Def We say that $f(z) \to 0$ uniformly as $R \to \infty$ in C_R if $|f(z)| \le k_R$, where k_R depends only on R (not on $\arg z$) and $k_R \to 0$ as $R \to \infty$



Jordan Inequality

Prove that $\frac{2\theta}{\pi} \le \sin\theta \le \theta \quad (0 \le \theta \le \pi/2)$

Proof # We know that as θ increases from 0 to $\pi/2$, $\cos\theta$ decreases steadily and consequently the mean ordinate of the graph of $y = \cos x$ over the range $0 \le x \le \theta$ also decreases steadily. But this mean ordinate is

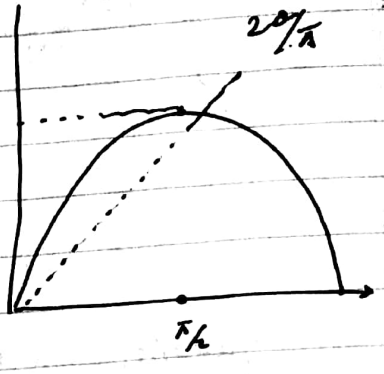
$$\frac{1}{\theta}\int_0^{\theta} \cos x\,dx = \frac{\sin\theta}{\theta}$$

It follows that when $0 \le \theta \le \pi/2$

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$$

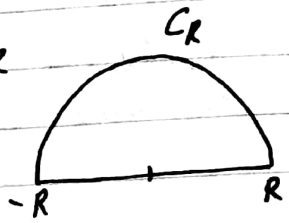
$$\Rightarrow \frac{2\theta}{\pi} \le \sin\theta \le \theta$$

This is known as Jordan's inequality



Jordan Lemma

Suppose that on the circular arc C_R we have $f(z) \to 0$ uniformly as $R \to \infty$



OR

$|f(z)| \le M/R^k$ for $z = Re^{i\theta}$, where $k > 0$ and M are constants

Then $\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)\,dz \to 0$ as $R \to \infty$, $m > 0$

$f(z)$ has no singularities other than poles in the upper half plane.

Proof $\because f(z) \to 0$ as $R \to \infty$

$\therefore |f(z)| \le K_R$ where K_R is constant and $K_R \to 0$ as $R \to \infty$

$$I = \left| \int_{C_R} e^{imz} f(z)\,dz \right|$$

$$\le \int_0^{\pi} e^{-my} K_R R\,d\theta$$

using $y = R\sin\theta$ & $\sin(\pi - \theta) = \sin\theta$

$$\int_0^{\pi} e^{-my}\,d\theta = \int_0^{\pi} e^{-mR\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta$$

But in the region $0 \le \theta \le \pi/2$, we have

$\sin\theta \ge 2\theta/\pi$ & hence $\int_0^{\pi} e^{-my}\,d\theta \le 2\int_0^{\pi/2} e^{-2mR\theta/\pi}$

So

$$I \le 2K_R R \int_0^{\pi/2} e^{-2mR\theta/\pi}\,d\theta$$

$$= 2K_R R \left| -\frac{e^{-2mR\theta/\pi}}{2mR/\pi} \right|_0^{\pi/2}$$

$$= \frac{2K_R R\pi}{2mR}\left(-e^{-mR} + e^{0}\right)$$

$$= \frac{K_R \pi}{m}\left(1 - e^{-mR}\right) \to 0 \text{ as } R \to \infty$$

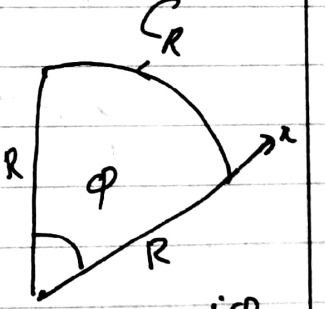
because $K_R \to 0$

Note # We note that if $m < 0$, a similar result holds for the contour of semi-circle. Moreover by suitably rotating the contour, Jordan's Lemma applies to the cases $m = il$ for $l \neq 0$. If $f = P/Q$ is a rational function, then Jordan's Lemma is satisfied if $\deg(Q)$ is one more than $\deg(P)$

Theorem # If on a circular arc C_R of radius R and centre $z = 0$, $z f(z) \to 0$ uniformly as $R \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

Proof # Let ϕ be the angle enclosed by the arc C_R. Then



$$\left| \int_{C_R} f(z)\,dz \right| \le \int_0^{\phi} |f(z)|\,|dz|$$

$$\le \int_0^{\phi} |f(z)|\,R\,d\theta$$

$$\le K_R \phi$$

$z = Re^{i\theta}$

$dz = iRe^{i\theta}d\theta$

$= iz\,d\theta$

$|dz| = R\,d\theta$

Because $z f(z) \to 0$ uniformly, it follows

that $|z f(z)| = |z|\,|f(z)| = R|f(z)| \leq k_R$

$k_R \to 0$ as $R \to \infty$

Thus $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$

Example # $I = \int_{-\infty}^{+\infty} \frac{\cos kx}{(x+b)^2 + a^2}$ $k \geq 0$, $a > 0$, b real

Sol # We consider the integral

$$I_+ = \int_{-\infty}^{+\infty} \frac{e^{ikz}}{(z+b)^2 + a^2}\,dz$$ + means in upper half

Poles of $f(z) = \frac{e^{ikz}}{(z+b)^2 + a^2}$ are roots of

$$(z+b)^2 + a^2 = 0$$
$$z + b = \pm ia$$
$$z = ia - b,\ -ia - b$$

only $z_1 = ia - b$ lies in the upper half plane because $a > 0$

$$\text{Res}(f, z_1) = \left[\frac{e^{ikz}}{2(z+b)}\right]_{z_1} = \frac{e^{ik(ai-b)}}{2ai}$$

$$\int_{-R}^{R} \frac{e^{ikz}\,dz}{(z+b)^2 + a^2} + \int_{C_R} \frac{e^{ikz}\,dz}{(z+b)^2 + a^2} = \frac{2\pi i \times e^{ik(ai-b)}}{2ai}$$

$$= \frac{\pi}{a} e^{-ka} \cdot e^{-ikb}$$

$$= \frac{\pi}{a} e^{-ka} (\cos kb - i \sin kb)$$

$$\left| \int_{C_R} \frac{e^{ikz}\,dz}{(z+b)^2 + a^2} \right| \leq \int_{C_R} \frac{|dz|}{|z|^2 - 2|b|\,|z| - a^2 - b^2}$$

$$= \frac{\pi R}{R^2 - 2bR - (a^2+b^2)} \xrightarrow[\text{as } R\to\infty]{} 0$$

Hence when $R \to \infty$, we have

$$I_+ = \int_{-\infty}^{+\infty} \frac{e^{ikx}\,dx}{(x+b)^2+a^2} = \frac{\pi}{a} e^{-ka}(\cos kb - i \sin kb)$$

$$\int_{-\infty}^{+\infty} \frac{\cos kx\,dx}{(x+b)^2+a^2} + i \int_{-\infty}^{\infty} \frac{\sin kx\,dx}{(x+b)^2+a^2}$$

$$= \frac{\pi}{a} e^{-ka}(\cos kb - i \sin kb)$$

Comparing we get

$$I = -\frac{\pi}{a} e^{-ka} \sin k \cdot \cos kb$$

$$J = \int_{-\infty}^{+\infty} \frac{\sin kx\,dx}{(x+b)^2+a^2} = -\frac{\pi}{a} e^{-ka} \sin bk$$

If $b = 0$ $J = 0$ which also follows directly from the fact that integrand is odd.

Example # Evaluate

$$I = 2\int_{-\infty}^{+\infty} \frac{x \sin\alpha x \cos\beta x}{x^2+\gamma^2}\,dx \qquad \gamma > 0, \alpha, \beta \text{ real}$$

Sol $\sin\alpha x \cos\beta x = \frac{1}{2}[\sin(\alpha-\beta)x + \sin(\alpha+\beta)x]$

$$I = \int_{-\infty}^{+\infty} \frac{x[\sin(\alpha-\beta)x + \sin(\alpha+\beta)x]\,dx}{x^2+\gamma^2}$$

This motivates to consider the integrals

$$J = \int_{-\infty}^{+\infty} \frac{x\, e^{i(\alpha-\beta)x}}{x^2+\gamma^2}\,dx + \int_{-\infty}^{+\infty} \frac{x\, e^{i(\alpha+\beta)x}\,dx}{x^2+\gamma^2}$$

$= J_1 + J_2$

Jordan's Lemma applies because the function.

$f(z) = \frac{z}{z^2 + r^2} \to 0$ uniformly and we note that

$$|f| \le \frac{R}{R^2 - r^2} \equiv k_R$$

we note that the denominator is only one degree higher than the numerator.

Case – I # If $\alpha - \beta > 0$, then we close our contour in the upper half plane and the only residue is $z = ir$ $(r > 0)$

$$\text{Res}\left(\frac{z e^{i(\alpha-\beta)z}}{z^2+r^2}, ir\right) = \lim_{z \to ir} \frac{z e^{i(\alpha-\beta)z}}{z + ir}$$

$$= \frac{ir \; e^{i(\alpha-\beta)ir}}{2ir}$$

$$= \frac{1}{2} e^{-(\alpha-\beta)r}$$

$$\text{Res}\left(\frac{z e^{i(\alpha+\beta)z}}{z^2+r^2}, +ir\right) = \lim_{z \to +ir} \frac{z e^{i(\alpha+\beta)z}}{z + ir}$$

$$= \frac{+ir}{+2ir} e^{i^2(\alpha+\beta)r}$$

$$= \frac{1}{2} e^{-(\alpha+\beta)r}$$

so

$$J_1 = 2\pi i \times \frac{1}{2} e^{-(\alpha-\beta)r} = i\pi e^{-(\alpha-\beta)r} \to (1)$$

$$J_2 = 2\pi i \times \frac{1}{2} e^{-(\alpha+\beta)r} = i\pi e^{-(\alpha+\beta)r} \to (4)$$

Case – II if $\alpha - \beta < 0$, we close in the lower half plane and find.

$$J_1 = -i\pi e^{(\alpha-\beta)r} \longrightarrow (2)$$

Combing (1) & (2)

$$J_1 = i\pi \, \text{sgn}(\alpha - \beta) \, e^{-|\alpha-\beta|r} \longrightarrow (3)$$

For J_2

Case-I If $\alpha+\beta > 0$, then we close in upper half plane and $z = +ir$ is the pole inside the contour.

$$\text{Res}\left(\frac{z e^{i(\alpha+\beta)z}}{z^2+r^2}, +ir\right) = \frac{1}{2} e^{-(\alpha+\beta)r}$$

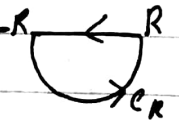
$$J_2 = 2\pi i \times \frac{1}{2} e^{-(\alpha+\beta)r} = i\pi e^{-(\alpha+\beta)r} \rightarrow (4)$$

Case-II # If $\alpha+\beta < 0$, we close in lower half plane and $z = -ir$ is the pole inside contour.

$$\text{Res}\left(\frac{z e^{i(\alpha+\beta)z}}{z^2+r^2}, -ir\right) = \lim_{z \to -ir} \frac{z e^{i(\alpha+\beta)z}}{z - ir}$$

$$= \frac{-ir}{-2ir} \cdot e^{(\alpha+\beta)r}$$

$$= \frac{1}{2} e^{(\alpha+\beta)r}$$



$$J_2 = -i\pi e^{(\alpha+\beta)r} \rightarrow (5)$$

Combining (4) & (5)

$$J_2 = i\pi \, \text{sgn}(\alpha+\beta) e^{-|\alpha+\beta|r} \rightarrow (6)$$

Thus

$$J = J_1 + J_2$$

$$= i\pi \left[\text{sgn}(\alpha-\beta) e^{-|\alpha-\beta|r} + \text{sgn}(\alpha+\beta) e^{-|\alpha+\beta|r} \right]$$

By taking the imaginary part.

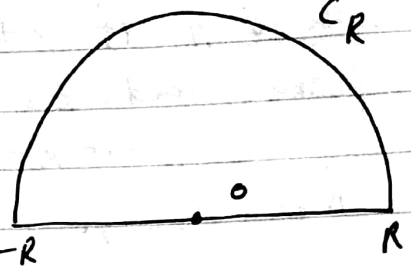
$$I = \pi \left[\text{sgn}(\alpha-\beta) e^{-|\alpha-\beta|r} + \text{sgn}(\alpha+\beta) e^{-|\alpha+\beta|r} \right]$$

If we take $\text{sgn}(0) = 0$ then the case $\alpha = \beta$ is incorporated in this result. This could either be established using $\sin\alpha x \cos\alpha x = \frac{1}{2}\sin 2\alpha x$ or by noting that $J_1 = 0$ owing to the oddness of the integrand.

$$\operatorname{sgn}(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$$

Example# Show that $\int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}$ $m>0$

Sol# Consider $\oint_C \frac{e^{imz}}{z^2+1}\,dz$ where C is the contour consisting of semi-circle C_R and x-axis from $-R$ to R



$$\left| \int_{C_R} \frac{e^{imz}}{z^2+1}\,dz \right|$$

$$\leq \int_{C_R} \frac{|dz|}{|z^2+1|} \leq \int_{C_R} \frac{|dz|}{|z|^2-1} = \frac{\pi R}{R^2-1}$$

$\to 0$ as $R\to\infty$

The integrand has simple poles at $z=\pm i$, but only $z=i$ lies within C

$$\text{Res at } z=i = \lim_{z\to i}\left\{\frac{(z-i)\,e^{imz}}{(z-i)(z+i)}\right\}$$

$$= e^{-m}/2i$$

Then $$\oint_C \frac{e^{imz}}{z^2+1}\,dz = 2\pi i\left(\frac{e^{-m}}{2i}\right) = \pi e^{-m}$$

$$\int_{-R}^{R} \frac{e^{imx}}{x^2+1}\,dx + \int_{C_R} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

when $R\to\infty$

$$\int_{-\infty}^{+\infty} \frac{e^{imx}}{x^2+1}\,dx + 0 = \pi e^{-m}$$

$$\int_{-\infty}^{+\infty} \frac{\cos mx}{x^2+1}\,dx + i\int_{-\infty}^{+\infty} \frac{\sin mx}{x^2+1}\,dx = \pi/e^m$$

$\Rightarrow$

Comparing

$$\int_{-\infty}^{+\infty} \frac{\cos mx}{x^2+1}\,dx = \pi e^{-m}$$

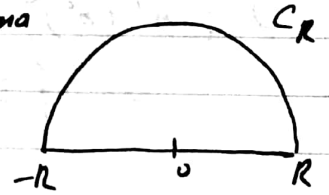
$$2\int_0^{\infty} \frac{\cos mx}{x^2+1}\,dx = \pi e^{-m}$$

$$\int_0^{\infty} \frac{\cos mx\,dx}{x^2+1} = \frac{\pi}{2} e^{-m}$$

Example # Show that if $b>0$, $m>0$, then

$$\int_0^{\infty} \frac{3x^2-a^2}{(x^2+b^2)^2} \cos mx\,dx = \frac{\pi e^{-mb}}{4b^3}\{3b^2 - a^2 - mb(3b^2+a^2)\}$$

Sol We consider $\int_C \frac{3z^2-a^2}{(z^2+b^2)^2} e^{miz}\,dz$, the Contour C being Semi-Circle C_R of x-axis from $-R$ to R on the Semi-Circle C_R, by Jordan's lemma, the integral tends to zero as $R\to\infty$



Hence

$$\int_{-\infty}^{+\infty} \frac{3x^2-a^2}{(x^2+b^2)^2} e^{mix}\,dx = 2\pi i \left(\text{Sum of the residues at poles in the upper half plane}\right)$$

The only pole inside the Contour is at $z_0 = ib$ of order 2.

Let $z - ib = z_1$

$$\frac{(3z^2-a^2)e^{miz}}{(z+ib)^2(z-ib)^2} = \frac{1}{z_1^2} \frac{e^{-mb}\, e^{miz_1}\{3(z_1+ib)^2 - a^2\}}{(z_1+2ib)^2}$$

Expanding and ignoring terms of order z_1^2 and above expression gives

$$-\frac{1}{z_1^2}\frac{e^{-mb}}{4b^2}\left(1+miz_1+\cdots\right)\left(-a^2-3b^2+6ibz_1+\cdots\right)\times\left(1+\frac{z_1}{2ib}\right)^{-2}$$

$$=-\frac{1}{z_1^2}\frac{e^{-mb}}{4b^2}\left[-(a^2+3b^2)+z_1\{6ib-im(a^2+3b^2)\}\right]\times\left[1-\frac{z_1}{ib}\right]$$

picking out the term in $\frac{1}{z_1}$ the residue is

$$\frac{e^{-mb}}{4b^2}\left\{\frac{i}{b}(a^2+3b^2)-6ib+mi(a^2+3b^2)\right\}$$

Hence

$$\int_0^\infty \frac{(3x^2-a^2)}{(x^2+b^2)^2}e^{mix}\,dx+\int_{-\infty}^{0}\frac{3x^2-a^2}{(x^2+b^2)^2}e^{mix}\,dx$$

$$=\frac{2\pi e^{-mb}}{4b^3}\left\{-(a^2+3b^2)+6b^2-mb(3b^2+a^2)\right\}$$

$$\therefore\int_0^\infty\frac{3x^2-a^2}{(x^2+b^2)^2}\cos mx\,dx$$

$$=\left\{3b^2-a^2-mb(3b^2+a^2)\right\}\frac{\pi e^{-mb}}{4b^3}$$

Example # Evaluate & prove that

$$\int_0^\infty\frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

Where $a>b>0$

Sol # Take $f(z)=\dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

and choose the closed Contour C as Consisting of the upper half C_R of a large Circle $|z|=R$ and the real axis $-R$ to R

Poles of $f(z)$ are given by

$(z^2+a^2)(z^2+b^2)=0 \qquad z = \pm ia, \pm ib$

of which the only poles lying inside C are $z=ai, z=ib$

Residue of $f(z)$ at $z=ai = \lim_{z\to ai}(z-ai)f(z)$

$$= \lim_{z\to ai} \frac{(z-ai)\, e^{iz}}{(z^2+a^2)(z^2+b^2)}$$

$$= \frac{e^{i\cdot ia}}{(ai+ai)(i^2a^2+b^2)}$$

$$= \frac{e^{-a}}{2ai(-a^2+b^2)} = -\frac{e^{-a}}{2ai(a^2-b^2)}$$

$$\text{Res}(f, ib) = \frac{e^{-b}}{2ib(a^2-b^2)}$$

By Cauchy's residue theorem

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum R^+$$

$$= \frac{2\pi i}{2i(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$= \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Where

$$\left|\int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz\right| \le \int_{C_R} \frac{|e^{iz}|\,|dz|}{(|z|^2-a^2)(|z|^2-b^2)}$$

$\because \quad |z_1+z_2| \ge |z_1| - |z_2|$

$$\leq \int_0^{\pi} \frac{|e^{-R\sin\theta}| \cdot R\,d\theta}{(R^2-a^2)(R^2-b^2)}$$

$z = Re^{i\theta}$ and $\oint_0 |dz| = R\,d\theta$

$$\leq \int_0^{\pi} \frac{e^{-R} R\,d\theta}{(R^2-a^2)(R^2-b^2)} = \frac{\pi R e^{-R}}{(R^2-a^2)(R^2-b^2)}$$

$\to 0$ as $R \to \infty$

$\therefore$ as $R \to \infty$, we have.

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

Equating real and imaginary parts we get

$$\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

and $$\int_{-\infty}^{+\infty} \frac{\sin x\,dx}{(x^2+a^2)(x^2+b^2)} = 0$$

Example # Prove that $\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx$

$$= 2\pi e^{-a} \quad (a > 0)$$

Sol # The Given Integral $\int_{-\infty}^{+\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx$ can be written as

$$\int_C \frac{a e^{iz} + z e^{iz}}{z^2+a^2}\,dz$$ where C is a Contour

Contains x-axis from $-R$ to R and semi-circle C_R in the upper half plane.

$$\int_C \frac{ae^{iz}+ze^{iz}}{z^2+a^2}dz = \int_C \frac{ae^{iz}dz}{z^2+a^2} + \int_C \frac{ze^{iz}dz}{z^2+a^2}$$

$\downarrow I_1$ $\qquad$ $\downarrow I_2$

The poles are at $z = \pm ai$ and $z = ai$ lies in the upper half plane

$$R_{I_1}(f, ai) = \lim_{z\to ai}\left[\frac{ae^{iz}}{z+ai}\right] = \frac{ae^{-a}}{2ai}$$

$$= \frac{e^{-a}}{2i}.$$

$$R_{I_2}(f, ai) = \lim_{z\to ai}\left[\frac{ze^{iz}}{z+ai}\right] = \frac{aie^{-a}}{2ai}$$

$$= \frac{e^{-a}}{2}.$$

$$\int_{-\infty}^{+\infty} \frac{a\cos x + x\sin x}{x^2+a^2}dx = Re\left[2\pi i \times \frac{e^{-a}}{2i}\right] + Im\left[2\pi i \cdot \frac{e^{-a}}{2}\right]$$

$$= \pi e^{-a} + \pi e^{-a} = 2\pi e^{-a}$$

Example # $\int_0^{\infty} \frac{\cos mx}{x^4+x^2+1}dx = \frac{\pi}{\sqrt{3}}\sin\left(\frac{m}{2}+\frac{\pi}{6}\right)e^{-\frac{m\sqrt{3}}{2}}$

Sol Consider the Integral $\int_C \frac{e^{imz}}{z^4+z^2+1}$, where C is a Contour Consisting of semi-circle C_R in the upper half plane & x-axis from $-R$ to R

Poles are roots of

$$z^4+z^2+1 = 0$$

$$(1-z^2)(z^4+z^2+1) = 0$$

$$\Rightarrow 1-z^6 = 0 \qquad z^6 = 1 = \cos 0 + i\sin 0 = e^{i0}$$

$$z_k = e^{\frac{2k\pi i}{6}}$$

poles are $e^{0}, e^{\pi i/3}, e^{2\pi i/3}, -1, e^{4\pi i/3}, e^{5\pi i/3}$

only $z_1 = e^{\pi i/3}$ $z_2 = e^{2\pi i/3}$ lie in the upper plane.

Let, $e^{\pi i/3} = \alpha$, then $\alpha^2 = e^{2\pi i/3}$

$$R(f, e^{\pi i/3} = \alpha) = \lim_{z \to \alpha} \left[\frac{e^{imz}(z-\alpha)}{z^4+z^2+1} \right]_{z=\alpha} \quad (\%)$$

$$= \lim_{z \to \alpha} \left[\frac{e^{imz} \cdot 1 + (z-\alpha) e^{imz}(im)}{4z^3+2z} \right]_{z=\alpha}$$

$$= \frac{e^{im\alpha}}{4\alpha^3+2\alpha}$$

$$R(f, \alpha^2) = \frac{e^{im\alpha^2}}{4\alpha^6+2\alpha^2}$$

Sum of the Residues $= \frac{e^{im\alpha}}{4\alpha^3+2\alpha} + \frac{e^{im\alpha^2}}{4\alpha^6+2\alpha^2}$

$$= \frac{e^{im\alpha}}{2(2\alpha^3+\alpha)} + \frac{e^{im\alpha^2}}{2(2\alpha^3+\alpha^2)}$$

$$= \frac{1}{2}\left[\frac{e^{im(1/2 + \frac{\sqrt{3}}{2}i)}}{2e^{i\pi} + e^{\pi i/3}} + \frac{e^{im(-1/2 + \sqrt{3}/2 i)}}{2e^{2\pi i} + e^{2\pi i/3}} \right]$$

$$= \frac{1}{2}\left[\frac{e^{mi/2} \cdot e^{-m\frac{\sqrt{3}}{2}i}}{-2 + 1/2 + \frac{\sqrt{3}}{2}i} + \frac{e^{-mi/2} \cdot e^{-m\frac{\sqrt{3}}{2}i}}{2 + (\frac{\sqrt{3}}{2}i - 1/2)} \right]$$

$$= \frac{1}{2} e^{-m\frac{\sqrt{3}}{2}i} \left[\frac{e^{mi/2}}{-3/2 + \frac{\sqrt{3}}{2}i} + \frac{e^{-mi/2}}{\frac{3}{2} + \frac{\sqrt{3}i}{2}} \right]$$

$$= e^{-m\frac{\sqrt{3}}{2}i} \left[\frac{e^{mi/2}}{\sqrt{3}i - 3} + \frac{e^{-mi/2}}{\sqrt{3}i + 3} \right]$$

$$= e^{-m\frac{\sqrt{3}}{2}i} \left[\frac{e^{mi/2}(\sqrt{3}i+3) + e^{-mi/2}(\sqrt{3}i-3)}{(\sqrt{3}i-3)(\sqrt{3}i+3)} \right]$$

$$= e^{-\frac{m\sqrt{3}i}{2}}\left[\frac{3\left(e^{\frac{mi}{2}}-e^{-\frac{mi}{2}}\right)+\sqrt{3}i\left(e^{mi/2}+e^{-mi/2}\right)}{-9-3}\right]$$

$$= e^{-\frac{m\sqrt{3}}{2}i}\left[\frac{3(2i\sin(m/2))+\sqrt{3}i\cdot 2\cos m/2}{-12}\right]$$

$$= -\frac{1}{12}e^{-\frac{m\sqrt{3}}{2}i}\left[6i\sin m/2+2\sqrt{3}i\cos m/2\right]$$

$$= -\frac{1}{12}e^{-\frac{m\sqrt{3}}{2}i}\left[6i\sin m/2+2\sqrt{3}i\cos m/2\right]$$

$$= -e^{-\frac{m\sqrt{3}i}{2}}\left(\frac{i}{2}\sin(m/2)+\frac{i}{2\sqrt{3}}\cos m/2\right)$$

$$= -\frac{i}{\sqrt{3}}e^{-\frac{m\sqrt{3}}{2}i}\left[\frac{\sqrt{3}}{2}\sin\frac{m}{2}+\frac{1}{2}\cos\frac{m}{2}\right]$$

$$= -\frac{i}{\sqrt{3}}e^{-\frac{m\sqrt{3}}{2}i}\left[\cos\pi/6\sin\frac{m}{2}+\sin\frac{\pi}{6}\cos\left(\frac{m}{2}\right)\right]$$

$$= -\frac{i}{\sqrt{3}}e^{-\frac{m\sqrt{3}}{2}i}\sin\left(\frac{m}{2}+\pi/6\right)$$

$$\int_{-\infty}^{+\infty}\frac{\cos mx\,dx}{x^4+x^2+1} = Re\left\{2\pi i\times\frac{-i}{\sqrt{3}}e^{-\frac{m\sqrt{3}}{2}i}\sin\left(\frac{m}{2}+\pi/6\right)\right.$$

$$= \frac{2\pi}{\sqrt{3}}e^{-\frac{m\sqrt{3}}{2}i}\sin\left(\frac{m}{2}+\pi/6\right)$$

$$\therefore \int_0^{\infty}\frac{\cos mx\,dx}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}}e^{-\frac{m\sqrt{3}}{2}i}\sin\left(\frac{m}{2}+\pi/6\right)$$

Example # If $a \geq 4$, then prove that

$$\int_0^{\infty}\frac{x\sin ax}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}}e^{-\frac{a\sqrt{3}}{2}}\sin\left(\frac{a}{2}\right)$$

Sol # The Given integral can be written as

$$\int_C\frac{z\,e^{iaz}\,dz}{z^4+z^2+1}$$

The poles are roots of $z^4+z^2+1=0$

poles are as in above example
$1, e^{\pi i/3}, e^{2\pi i/3}, e^{\pi i}, e^{4\pi i/3}, e^{5\pi i/3}$
The only poles which lie in the upper half plane are at
Let $z = \alpha = e^{\pi i/3}$ $\alpha^2 = e^{2\pi i/3}$

$$R(f, \alpha) = \lim_{z \to \alpha} \left[\frac{z(z-\alpha) e^{iaz}}{z^4 + z^2 + 1} \right] \quad (\%)$$

$$= \lim_{z \to \alpha} \left[\frac{ze^{iaz} + (z-\alpha) e^{iaz} + z(z-\alpha) e^{aiz} ia}{4z^3 + 2z} \right]$$

$$= \frac{\alpha e^{ia\alpha}}{2\alpha(1+2\alpha^2)} = \frac{e^{ia\alpha}}{2+4\alpha^2}$$

$$R(f, \alpha^2) = \frac{\alpha^2 e^{ia\alpha^2}}{2\alpha^2(1+2\alpha^4)} = \frac{e^{ia\alpha^2}}{2+4\alpha^4}$$

$$\text{Sum of Residues} = \frac{e^{ia\alpha}}{2+4\alpha^2} + \frac{e^{ia\alpha^2}}{2+4\alpha^4}$$

$$= \frac{e^{ia e^{\pi i/3}}}{2+4e^{2\pi i/3}} + \frac{e^{ia\frac{2\pi i}{3}}}{2+4e^{\frac{4\pi i}{3}}}$$

$$= \frac{e^{ia(1/2 + i\frac{\sqrt{3}}{2})}}{2+4(-1/2+\frac{\sqrt{3}}{2}i)} + \frac{e^{ai(-1/2+i\frac{\sqrt{3}}{2})}}{2+4(-1/2-\frac{\sqrt{3}}{2}i)}$$

$$= \frac{e^{-a\sqrt{3}/2} \cdot e^{ai/2}}{2\sqrt{3}\,i} + \frac{e^{-a\sqrt{3}/2} \cdot e^{-ai/2}}{-2\sqrt{3}\,i}$$

$$= \frac{e^{-a\sqrt{3}/2}}{2\sqrt{3}\,i} \left[e^{\frac{a}{2}i} - e^{-\frac{a}{2}i} \right]$$

$$= \frac{e^{-a\sqrt{3}/2}}{2\sqrt{3}\,i} (2i \sin a/2) = \frac{e^{-a\sqrt{3}/2}}{\sqrt{3}} \sin(a/2)$$

$$\int_{-\infty}^{\infty} \frac{x \sin a\, dx}{x^4+x^2+1} \overset{71}{=} Im \frac{2\pi i \times e^{-a\sqrt{3}/2} \sin(a/2)}{\sqrt{3}}$$

$$= \frac{2\pi}{\sqrt{3}} e^{\frac{-a\sqrt{3}}{2}} \sin(a/2)$$

$$\int_{0}^{\infty} \frac{x \sin a\, dx}{x^4+x^2+1} = \frac{\pi}{\sqrt{3}} e^{\frac{-a\sqrt{3}}{2}} \sin(a/2)$$

Example # prove that

$$\int_0^{\infty} \frac{\cos mx}{x^4+a^4} = \frac{\pi}{2a^2} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

Sol Consider the integral

$$\int_C \frac{\cancel{\cos mz}\, e^{imz}}{z^4+a^4} dz$$

The poles are roots of $z^4+a^4=0$

$$z^4 = -a^4 = a^4 e^{i\pi} = a^4 e^{(2k\pi+\pi)i}$$

$$z_k = a\, e^{\left(\frac{2k\pi+\pi}{4}\right)i} \qquad k = 0,1,2,3$$

$$z_0 = a e^{\pi/4 i} = \frac{a}{\sqrt{2}}(1+i)$$

$$z_1 = a e^{3\pi i/4} = \frac{a}{\sqrt{2}}(-1+i)$$

These are only poles which lie in the upper half plane

Let α be any pole in the upper half plane.

$$R(f,\alpha) = \lim_{z\to\alpha} \left[\frac{e^{imz}(z-\alpha)}{z^4+a^4}\right] \qquad (\%)$$

$$= \lim_{z\to\alpha} \left[\frac{e^{imz} + (z-\alpha)\, e^{imz}(im)}{4z^3}\right] = \frac{e^{im\alpha}}{4\alpha^3}$$

$$= \frac{\alpha \cdot e^{im\alpha}}{4\alpha^4} = \frac{e^{im\alpha}\cdot\alpha}{-4a^4} \qquad z^4 = \alpha^4 = -a^4$$

$$= -\frac{1}{4a^4}\left[\alpha e^{imx}\right]$$

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$$R\left(f, \frac{a}{\sqrt{2}}(1+i)\right) = -\frac{1}{4a^4}\left[\frac{a}{\sqrt{2}}(1+i)\, e^{im\left(\frac{a}{\sqrt{2}}\right)(1+i)}\right]$$

$$R\left(f, \frac{a}{\sqrt{2}}(-1+i)\right) = -\frac{1}{4a^4}\left[\frac{a}{\sqrt{2}}(-1+i)\, e^{im\left(\frac{a}{\sqrt{2}}\right)(-1+i)}\right]$$

Sum of Residues

$$= -\frac{1}{4a^4}\left[\frac{a}{\sqrt{2}}(1+i)\, e^{-\frac{ma}{\sqrt{2}}} \cdot e^{\frac{mai}{\sqrt{2}}} + \frac{a}{\sqrt{2}}(-1+i)\, e^{-\frac{ma}{\sqrt{2}}} e^{-\frac{ima}{\sqrt{2}}}\right]$$

$$= -\frac{e^{-\frac{ma}{\sqrt{2}}}}{4a^3}\left[\frac{1+i}{\sqrt{2}} \cdot e^{\frac{ma}{\sqrt{2}}i} + \frac{1}{2}(-1+i)\, e^{-\frac{ima}{\sqrt{2}}}\right]$$

$$= -\frac{e^{-\frac{ma}{\sqrt{2}}}}{4a^3}\left[\frac{i}{\sqrt{2}}\left(e^{\frac{mai}{\sqrt{2}}} + e^{-\frac{mia}{\sqrt{2}}}\right) + \frac{1}{\sqrt{2}}\left(e^{\frac{mai}{\sqrt{2}}} - e^{-\frac{mai}{\sqrt{2}}}\right)\right]$$

$$= -\frac{e^{-\frac{ma}{\sqrt{2}}}}{4a^3}\left\{\frac{i}{\sqrt{2}} \cdot 2\cos\left(\frac{ma}{\sqrt{2}}\right) + \frac{2i}{\sqrt{2}}\sin\left(\frac{ma}{\sqrt{2}}\right)\right\}$$

$$= \frac{-2ie^{-\frac{ma}{\sqrt{2}}}}{4a^3}\left[\sin\pi/4\cos\left(\frac{ma}{\sqrt{2}}\right) + \cos\pi/4\sin\left(\frac{ma}{\sqrt{2}}\right)\right]$$

$$= \frac{-ie^{-\frac{ma}{\sqrt{2}}}}{2a^3}\sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

$$\int_{-\infty}^{+\infty} \frac{\cos mx\, dx}{x^4+a^4} = \mathrm{Re}\left[2\pi i \times \frac{-ie^{-\frac{ma}{\sqrt{2}}}}{2a^3}\sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)\right]$$

$$\int_0^{\infty} \frac{\cos dx}{x^4+a^4} = \frac{\pi}{2a^3}\, e^{-\frac{ma}{\sqrt{2}}}\sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

Z3

Example # prove that $\int_0^\infty \frac{x \sin x\, dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6e^2}(e-1)$

Sol Consider the integral

$$\int_C \frac{z e^{iz} dz}{(z^2+1)(z^2+4)}$$

where C is semi-circle C_R in the upper half plane and x-axis from $-R$ to R

The poles are at $z = \pm i \quad z = \pm 2i$

The poles $i, 2i$ lie in the upper half plane.

$$R(f, i) = \lim_{z \to i} \left[\frac{(z-i)\, z e^{iz}}{(z-i)(z^2+4)(z+i)} \right]$$

$$= \lim_{z \to i} \frac{z e^{iz}}{(z+i)(z^2+4)} = \frac{i e^{-1}}{2i \cdot 3} = \frac{1}{6e}$$

$$R(f\ 2i) = \lim_{z \to 2i} \left[\frac{z e^{iz} (z-2i)}{(z^2+1)(z-2i)(z+2i)} \right]$$

$$= \lim_{z \to 2i} \frac{z e^{iz}}{(z+2i)(z^2+1)}$$

$$= \frac{2i e^{-2}}{4i \times -3} = -\frac{1}{6e^2}$$

Sum of Residues $= \frac{1}{6e} - \frac{1}{6e^2} = \frac{1}{6e^2}(e-1)$

Two Important Results

The following two results are often of use in evaluating contour integrals

Theorem #A: Let C be the arc $\theta_1 \le \arg(z-a) \le \theta_2$ of the circle $|z-a| = R$ and if

$$\lim_{z \to a} (z-a) f(z) = k, \text{ where } k \text{ is constant}$$

, then

$$\lim_{r\to 0}\int_C f(z)\,dz = ik(\theta_2-\theta_1)$$

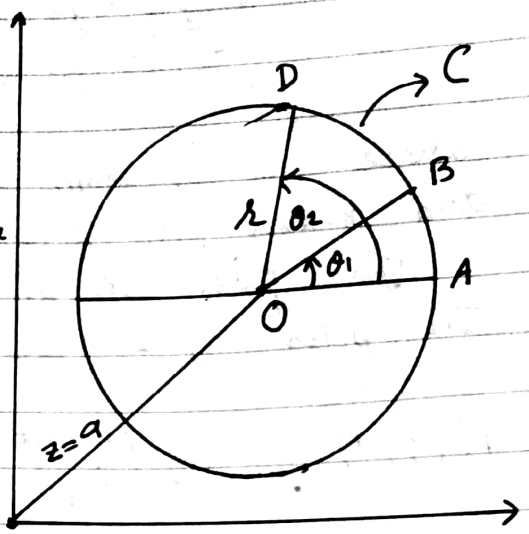
Proof #

Let $F(z)=(z-a)f(z)$

Since $f(z)$ & $z-a$ are individually continuous, Therefore their product $(z-a)f(z)$ must also be continuous.

For a given $\epsilon>0$, $\eta(\epsilon)$ can be found such that if $|z-a|<\eta$

$$|F(z)-F(a)|<\epsilon \qquad |z-a|<\eta$$



We can find $|\delta|<\epsilon$ such that

$$|F(z)-F(a)| = \delta < \epsilon$$

$$|F(z)-k| = \delta \qquad \lim_{z\to a}(z-a)f(z) = k = F(a)$$

$$F(z)-k = \pm\delta$$

$$F(z) = k\pm\delta$$

Let us now take a contour whose centre at $z=a$ and radius is r and $f(z)$ along the curve C between B & D.

$$\int_{BD} f(z)\,dz = \int_{BD}\frac{F(z)}{z-a}\,dz = \int_{BD}\frac{k\pm\delta}{z-a}\,dz$$

Let us assume that $z-a=re^{i\theta}$

$$dz = ire^{i\theta}d\theta$$

$$\frac{dz}{re^{i\theta}} = i\,d\theta$$

$$\frac{dz}{z-a} = i\,d\theta$$

$$\int_{BD} f(z)\,dz = i\int_{\theta_1}^{\theta_2} (k \pm \delta)\,d\theta$$

$$= i(\theta_2 - \theta_1)(k \pm \delta)$$

$$= ik(\theta_2 - \theta_1) \pm i\delta(\theta_2 - \theta_1)$$

Now $r \to 0$ as $z \to a$

$\because |F(z) - F(a)| = \delta \qquad z - a = re^{i\theta}$

$F(a) = k$

and $z \to a$ as $\delta \to 0$

$\therefore F(z) = k \pm \delta - F(a) \pm \delta$

Conclusively when $r \to 0$, $\delta \to 0$

Hence

$$\lim_{r \to 0} \int_{BD} f(z)\,dz = ik(\theta_2 - \theta_1)$$

OR

$$\int_C f(z)\,dz = \int_{BD} f(z)\,dz = i\int_C (k \pm \delta)\,d\theta$$

$$= ik(\theta_2 - \theta_1) \pm i\delta(\theta_2 - \theta_1)$$

$$\int_C f(z)\,dz - ik(\theta_2 - \theta_1) = \pm i\delta(\theta_2 - \theta_1)$$

$$\left| \int_C f(z)\,dz - ik(\theta_2 - \theta_1) \right| = |\delta|\,|\theta_2 - \theta_1|$$

$$< \eta\,|\theta_2 - \theta_1|$$

Since η can be made as small as we please

$\left| \int_C f(z)\,dz - ik(\theta_2 - \theta_1) \right|$ can be made as small as we please i.e

i.e

$$\lim_{r \to 0} \int_C f(z)\,dz = ik(\theta_2 - \theta_1)$$

Theorem # B If $\lim_{R \to \infty} z f(z) = k$, where k is a constant, Then

$$\lim_{R \to \infty} \int_C f(z)\,dz = ik(\theta_2 - \theta_1)$$

where the integral is taken along the arce

$$\theta_1 \leq \arg z_1 \leq \theta_2$$

of circle $|z| = R$

Proof # The proof is similar to that of Theorem (A)

Corllary # If $\theta_1 = 0$, $\theta_2 = 2\pi$ arc C covers the whole circumference

Type:-

Functions (Integrand) With Real Singularity (Pole) of Indentation of Contours

In the previous types the condition was that the integrand has no pole on the real axis, Frequently in application

we are also interested in integrals with integrands that have singularities at finite location.

Consider the integral $\int_a^b f(x)\,dx$, where has a singularity at x_0

$$a < x_0 < b.$$

Convergence of such an integral depends on the existence of the following limit.

$$I = \int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{x_0-\epsilon} f(x)\,dx + \lim_{\delta \to 0^+} \int_{\delta+x_0}^b f(x)\,dx \quad \rightarrow (A)$$

where both ϵ & δ approach zero independtly and through +ve values.

It may happen that o' neigther of limits exists if $\epsilon, \eta \to 0$ independently but

$$\lim_{\epsilon \to 0} \left[\int_a^{x_0-\epsilon} f(x)\,dx + \lim \int_{x_0+\epsilon}^b f(x)\,dx \right] \quad \rightarrow (B)$$

exists. This is called the Cauchy Principal value of the integral. It is written as

$$P.V. \int_a^b f(x)\,dx.$$

We say that the Cauchy Principal Value integral exists iff the limit (B) exists.

For example the integral

$$\int_{-1}^{2} \frac{1}{x}\,dx = \lim_{\epsilon \to 0^+} \int_{-1}^{0-\epsilon} \frac{1}{x}\,dx + \lim_{\delta \to 0} \int_{0+\delta}^{2} \frac{1}{x}\,dx$$

$$= \lim_{\epsilon \to 0^+} \ln|\epsilon| - \lim_{\delta \to 0^+} |\delta| + \ln 2$$

does not exist, whereas

$$P.V. \int_{-1}^{2} \frac{1}{x}\,dx = \lim_{\epsilon \to 0} (\ln|\epsilon| - \ln|\epsilon|) + \ln 2 = \ln 2$$

The principal value exists although the integral itself has no meaning.

General Case

More generally, in applications we are sometimes interested in functions on an infinite interval with many points $\{x_i\}_{i=1}^{N}$ for which $\lim_{x \to x_i} f(x)$ is either infinite or does not exist.

We say the following Cauchy principal value integral exists iff for $a < x_1 < x_2 < \cdots < x_N < b$

$$P.V. \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R \to \infty} \left(\int_{-R}^{a} + \int_{b}^{R} \right) f(x)\,dx$$

$$+ \lim_{\epsilon_1, \epsilon_2 \cdots \epsilon_N \to 0^+} \left(\int_{a}^{x_1 - \epsilon_1} + \int_{x_1 + \epsilon_1}^{x_2 - \epsilon_2} + \int_{x_2 + \epsilon_2}^{x_3 - \epsilon_3} + \cdots + \int_{x_{N-1} + \epsilon_{N-1}}^{x_N - \epsilon_N} + \int_{x_N + \epsilon_N}^{b} \right) f(x)\,dx \quad \rightarrow (c)$$

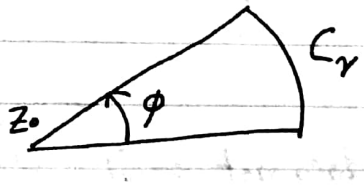
exists

In practice we usually combine integrals and consider the double limit $R \to \infty$ and $\epsilon_i \to 0^+$, for example

$\int_{-R}^{x_1 - \epsilon_1}$ in (c) and do not bother to partition the integrals into intermediate with a, b inserted.

The following Theorems are very useful

Theorem # Consider integrals a small circular arc with radius R, centre at $z = z_0$ and with the arc subtending an angle φ. Then



(a) If $(z - z_0) f(z) \to 0$ uniformly (independent of angle θ along C_r) as $r \to 0$. Then $\lim_{r \to 0} \int_{C_r} f(z)\,dz = 0$

(b) # Suppose $f(z)$ has a simple pole at $z=z_0$. Then for the Contour C_r

$$\lim_{r\to 0}\int_{C_r} f(z)dz = i\varphi \operatorname*{Res}_{z=z_0} f(z)$$

Proof # (a) Since $(z-z_0)f(z)\to 0$ uniformly as $r\to 0$, on C_r

$$|(z-z_0)f(z)| \le k_r, \text{ where } k_r$$

depends on r, not on $\arg(z-z_0)$ and $k_r\to 0$ as $r\to 0$.

So

$$|(z-z_0)f(z)| \le k_r$$

$$|z-z_0||f(z)| \le k_r$$

$$|f(z)| \le \frac{k_r}{|z-z_0|} = \frac{k_r}{r}$$

Let $z-z_0 = re^{i\theta}$ $\quad dz = ire^{i\theta}d\theta$

$$\left|\int_{C_r} f(z)dz\right| \le \int_{C_r}|f(z)||dz|$$

$$\le \frac{k_r}{r}\int_0^{\varphi} r\,d\theta = k_r\varphi \to 0 \text{ as } r\to 0$$

$$\Rightarrow \lim_{r\to 0}\int_{C_r} f(z)dz = 0$$

(b) If $f(z)$ has a simple pole with $\operatorname{Res}(f, z_0)=b_1$, Then from Laurent expansion in the neighbourhood of $z=z_0$

$$f(z) = \frac{b_1}{z-z_0} + g(z)$$

where $g(z)$ is analytic in the neighbourhood of $z=z_0$ or on the arc of integration

$$\lim_{r\to 0}\int_{C_r} f(z)\,dz = \lim_{r\to 0}\int_{C_r}\frac{b_1}{z-z_0}dz + \lim_{r\to 0}\int_{C_r} g(z)dz$$

Now
$$I_1 = \int_{C_r}\frac{b_1}{z-z_0}dz = b_1\int_0^{\phi}\frac{ire^{i\theta}d\theta}{re^{i\theta}} \qquad \text{Let } z-z_0 = re^{i\theta},\ dz = ire^{i\theta}d\theta$$
$$= b_1 i\int_0^{\phi} d\theta = i\phi b_1$$

$$I_2 = \int_{C_r} g(z)\,dz$$

Here $|g(z)| \le M$ = Constant in the neighbourhood of $z = z_0$, because $g(z)$ is analytic there

$$|I_2| = \left|\int_{C_r} g(z)dz\right|$$
$$\le \int_{C_r}|g(z)|\,|dz| \le M\int_{C_r}|dz| = M\phi\, r\to 0 \text{ as } r\to 0$$

$$\Rightarrow \quad I_2 = \int_{C_r} g(z)dz = 0$$

Thus $$\lim_{r\to 0}\int_{C_r} f(z)dz = i\phi b_1 = i\phi\,\text{Res}(f, z_0)$$

Evaluation of Integrals With Real Poles

To evaluate the integrals whose integrand has poles on the real axis we use the small-semi-circles to avoid these singularities. The following Theorem is helpful.

Theorem # Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function with a real pole at $z = a$. Let the semi-circle indenting this pole be C_ϵ with centre at $z = a$ and radius ϵ. If C_R is semi-circle sufficiently large to enclose all the poles of $f(z)$ in the upper half plane. Then

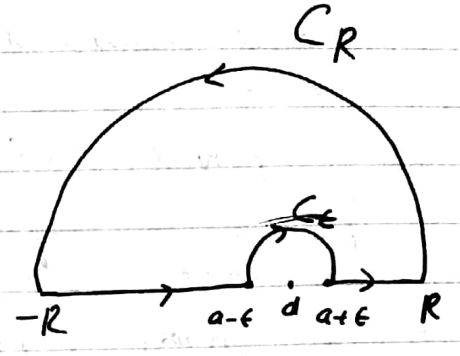
Here degree of $Q(z)$ at least two units higher than $P(z)$

$$P.V. \int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum R^{+} + \pi i \sum Res_{0}$$

Where $\sum R^{+}$ = Sum of residues in the upper plane

$\sum R_0$ = Sum of residues at poles on real line

Proof# Let $f(z)$ has only one pole on real line say at $z=a$ and C_ϵ is semi-circle centred at $z=a$ and of radius ϵ.



Let C be the contour consisting of C_R, C_ϵ, x-axis from $-R$ to $a-\epsilon$ and from $a+\epsilon$ to R. Then pole at $z=a$ is avoided and we have

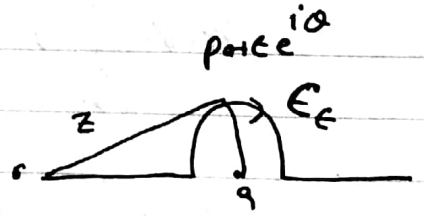
$$\int_C f(z)\,dz = 2\pi i \,(\text{Sum of Residues in the upper half plane})$$

$$\int_{-R}^{a-\epsilon} f(x)\,dx + \int_{C_\epsilon} f(z)\,dz + \int_{a+\epsilon}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\,(\text{Sum of residues in upper plane})$$

Here $z f(z) \to 0$ as $z \to \infty$ & $R \to \infty$

$$\therefore \int_{C_R} f(z)\,dz = 0$$

Consider $\int_{C_\epsilon} f(z)\,dz$



Let $z = a + \epsilon e^{i\theta}$ where $z=a$ when $\epsilon \to 0$

Then $dz = i\epsilon e^{i\theta} d\theta = i(z-a)\,d\theta$

$$\int_{C_\epsilon} f(z)\,dz = i \int_{\pi}^{0} f(z)(z-a)\,d\theta$$

$$\therefore \lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)\,dz = i \int_{\pi}^{0} \lim_{z \to a} (z-a) f(z)\,d\theta$$

$$= i\,\mathrm{Res}(f, a) \int_{\pi}^{0} d\theta$$

$$= i \sum R_0 (-\pi) = -i\pi \sum R_0$$

where R_0 is the residue at $z = a$, the simple pole, on real axis. Hence

$$\int_{-R}^{a-\epsilon} f(u)\,du - i\pi \sum R_0 + \int_{a+\epsilon}^{R} f(u)\,du = 2\pi i \sum R^+$$

$$\Rightarrow \int_{-R}^{a-\epsilon} f(u)\,du + \int_{a+\epsilon}^{R} f(u)\,du = 2\pi i \sum R^+ + i\pi \sum R_0$$

where $R \to \infty$

$$\lim_{R \to \infty} \left(\int_{-R}^{a-\epsilon} + \int_{a+\epsilon}^{R} \right) f(u)\,du = 2\pi i \sum R^+ + i\pi \sum R_0$$

$$\left(\int_{-\infty}^{a-\epsilon} + \int_{a+\epsilon}^{\infty} \right) f(u)\,du = 2\pi i \sum R^+ + i\pi \sum R_0$$

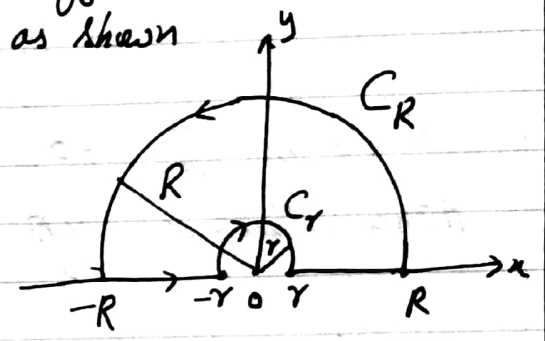
$$\lim_{\epsilon \to 0} \left(\int_{-\infty}^{a-\epsilon} + \int_{a+\epsilon}^{\infty} \right) f(u)\,du = 2\pi i \sum R^+ + i\pi \sum R_0$$

$$P.V. \int_{-\infty}^{+\infty} f(u)\,du = 2\pi i \sum R^+ + i\pi \sum R_0$$

Important Note # There is no special merit in using a particular curve as Contour for a particular integration, but in fact any circles, Semi-circle, quadrant of a circle or a rectangle whichever is suitable can be used unless otherwise stated. e.g. If the integrant is periodic function, then rectangular Contour is rather convenient

Example # Prove that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Sol # We do not consider $\frac{\sin z}{z}$ because this function does not behave suitably at infinity. We consider function $\frac{e^{iz}}{z}$ which has a simple pole at $z=0$. Since we can not integrate through a singularity, we modify the contour by indenting the path at $z=0$ as shown



Now the function $\frac{e^{iz}}{z}$ is integrated round the contour C consisting of

(i) The x-axis from r to R
(ii) the upper half of the circle $|z| = R$
(iii) the x-axis from $-R$ to $-r$
(iv) the upper half of the circle $|z| = r$

Since $f(z) = \frac{e^{iz}}{z}$ is analytic on and within the contour C, by Cauchy theorem

$$\int_C f(z)\,dz = 0$$

$$\Rightarrow \int_{-R}^{-r} f(z)\,dz + \int_{C_r} f(z)\,dz + \int_r^R f(z)\,dz + \int_{C_R} f(z)\,dz = 0$$

$$I_1 + I_2 + I_3 + I_4 = 0 \rightarrow (1)$$

$$I_4 = \int_{C_R} f(z)\,dz$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq \int_{C_R} |f(z)|\,|dz| = \int_{C_R} \frac{|e^{iz}|}{|z|}\,|dz|$$

$$\left| \int_{C_R} f(z)\,dz \right| \le \int_{C_R} \frac{|e^{iz}|\,|dz|}{|z|}$$

$$= \int_0^{\pi} \frac{e^{-R\sin\theta}}{R} \cdot R\,d\theta \qquad z = Re^{i\theta},\ |dz| = R\,d\theta$$

$$= \int_0^{\pi} e^{-R\sin\theta}\,d\theta$$

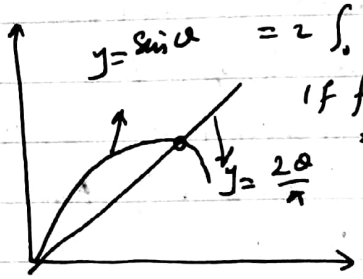
$$= 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta$$

$|e^{iz}| = |e^{i(R\cos\theta + iR\sin\theta)}| = |e^{iR\cos\theta}|\,|e^{-R\sin\theta}| = e^{-R\sin\theta}$

$\int_0^{2a} f(u)\,du = 2\int_0^{a} f(u)\,du$ if $f(2a-u) = f(u)$

$\sin\theta \ge \frac{2\theta}{\pi}$ if $0 \le \theta \le \pi/2$

Hence $-\sin\theta \le -\frac{2\theta}{\pi}$



$$|I_4| \le 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \le 2\int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}}\,d\theta$$

$$= 2\left(\frac{-\pi}{2R}\right)\left[e^{\frac{-2R\theta}{\pi}}\right]_0^{\pi/2}$$

$$= -\frac{\pi}{R}\left[1 - e^{-R}\right] \to 0 \text{ as } R \to \infty$$

<u>Integral</u> $I_2 = \int_{C_r} f(z)\,dz$

Let $z = re^{i\theta}$ $\quad dz = ire^{i\theta}\,d\theta = iz\,d\theta$

$$\int_{C_r} f(z)\,dz = \int_{C_r} \frac{e^{iz}}{z}\,iz\,d\theta = -\int_{\pi}^{0} ie^{ire^{i\theta}}\,d\theta$$

(clockwise)

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = -\int_{\pi}^{0} i\,d\theta = \pi i$$

OR

Residue at $z=0$ of $f(z) = \lim_{z\to 0} (z-0) f(z)$

$$= \lim_{z\to 0} z \cdot \frac{e^{iz}}{z}$$

$$= 1$$

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = -i\,(\text{Central angle of area } C_r)\,\text{Res}_{z=0} f$$

$$= -i\,\pi \cdot 1 = -i\pi$$

(-ve sign is taken C_r is clockwise direction)

Using in ①

$$+\int_{-R}^{-r} f(z)\,dz - i\pi + \int_{r}^{R} f(z)\,dz + 0 = 0$$

Taking limit when $R\to\infty \quad r\to 0$

$$\int_{-\infty}^{+\infty} \frac{e^{iz}}{z}\,dz = \pi i$$

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx = \pi i \qquad f(x) = \frac{e^{ix}}{x} = \frac{\cos x + i\sin x}{x}$$

$$f(-x) = \frac{\cos x - i\sin x}{-x}$$

$$\int_{-\infty}^{0} \frac{e^{ix}}{x}\,dx + \int_{0}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i$$

$$I_1 \int_{-\infty}^{0} \frac{e^{ix}}{x}\,dx$$

let $x = -t$ when $x=-\infty \quad t=\infty$

$dx = -dt \qquad x=0 \quad t=0$

$$\int_{-\infty}^{0} \frac{e^{ix}}{x}\,dx = -\int_{\infty}^{0} \frac{e^{-it}}{-t}\,dt = \int_{\infty}^{0} \frac{e^{-it}}{t}\,dt$$

$$= -\int_{0}^{\infty} \frac{e^{-ix}}{x}\,dx$$

$$\Rightarrow \int_{0}^{\infty} \frac{e^{ix} - e^{-ix}}{x}\,dx = \pi i \Rightarrow 2i\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi i$$

$$\int_{0}^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$$

OR

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{C_r} \frac{e^{iz}}{z}\,dx + \int_{r}^{R} \frac{e^{ix}}{x}\,dz + \int_{C_R} \frac{e^{iz}}{z}\,dz = 0$$

Replacing x by $-x$ in 1st integral

$$\int_{r}^{R} \frac{e^{iz} - e^{-ix}}{x}\,dx = -\int_{C_r} \frac{e^{iz}}{z}\,dx + \int_{C_R} \frac{e^{iz}}{z}\,dz$$

$$2i \int_{r}^{R} \frac{\sin x}{x}\,dx = +\pi i + 0$$

$$\int_{r}^{R} \frac{\sin x}{x}\,dx = \pi/2$$

$$\lim_{\substack{R\to\infty \\ r\to 0}} \int_{r}^{R} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

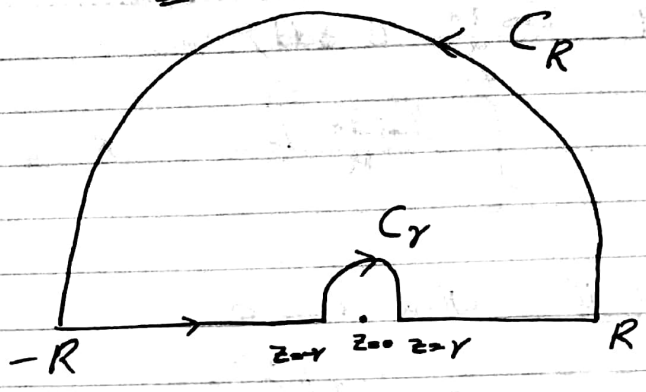
Example # $$\int_{-\infty}^{\infty} \frac{\sin ax}{x}\,dx = \operatorname{sgn}(a)\,\pi$$

where a is real &

$$\operatorname{sgn}(a) = \begin{cases} -1 & a < 0 \\ 0 & a = 0 \\ 1 & a > 0 \end{cases}$$

Sol # Case I First we consider $a > 0$ and contour depicted in the fig.

Consider $$\int_{-\infty}^{+\infty} \frac{e^{iaz}}{z}\,dz$$



Because there are no poles enclosed by the contour we have

$$\oint_C \frac{e^{iaz}}{z} dz = \left(\int_{-R}^{-r} + \int_{r}^{R} \right) \frac{e^{iax}}{x} dx + \int_{C_r} \frac{e^{iaz}}{z} dz + \int_{C_R} \frac{e^{iaz}}{z} dz$$

$$= 0$$

Because $a > 0$, the integral $\int_{C_R} \frac{e^{iaz}}{z} dz$ satisfies Jordan Lemma, hence it vanishes as $R \to \infty$.

$$\int_{C_r} \frac{e^{iaz}}{z} = -i\pi \, Res\left(\frac{e^{iaz}}{z}, 0\right)$$

$$= -i\bar{\pi} \cdot 1 = -i\bar{\pi}$$

The minus sign is a result of the direction being clockwise.

$$\left(\int_{-R}^{-r} + \int_{r}^{R} \right) \frac{e^{iax}}{x} dx - i\bar{\pi} = 0$$

Taking limit when $R \to \infty$, $r = 0$

$$P \int_{-\infty}^{\infty} \frac{\cos ax}{x} dx + iP \int_{-\infty}^{\infty} \frac{\sin ax}{x} dx = i\pi$$

Equating real and imaginary parts we obtain $P \int \frac{\cos ax}{x} dx = 0$ which is consistent with the fact

that $\frac{\cos ax}{x}$ is odd)

$$\int_{-\infty}^{+\infty} \frac{\sin ax}{x}\,dx = \pi = \operatorname{sgn}(a)\,\pi$$

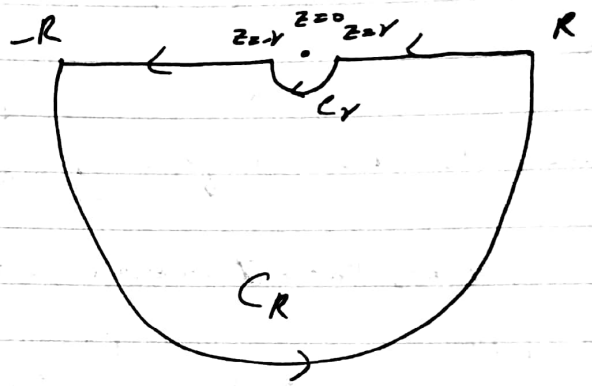
Case II When $a < 0$

$$I = \int_{-\infty}^{+\infty} \frac{\cos ax}{x}\,dx + i\int_{-\infty}^{+\infty} \frac{\sin ax}{x}\,dx = -I(-a) = i\pi$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\sin ax}{x} = \pi$$

$$2I = 0 \Rightarrow I = 0$$

The same result can be obtained by using the contour



Case III When $a = 0$

$$\int_{-\infty}^{+\infty} \frac{\sin ax}{x}\,dx = 0$$

Thus $$\int_{-\infty}^{+\infty} \frac{\sin ax}{x}\,dx = \operatorname{sgn}(a)\,\pi$$

Example Evaluate $I = \int_{-\infty}^{+\infty} \frac{\cos x - \cos a}{x^2 - a^2}$ a real

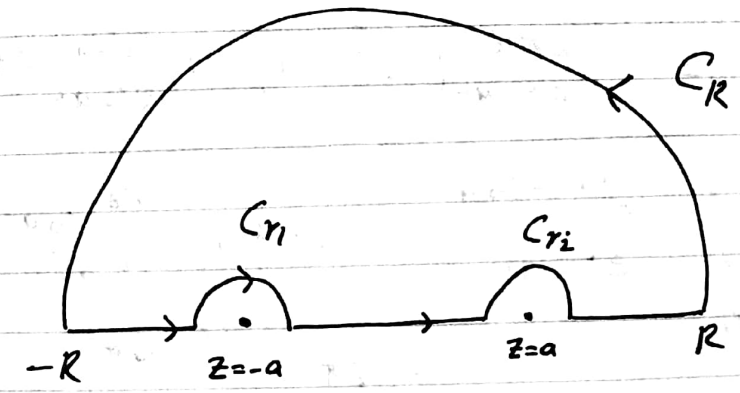
Sol : we note that the integral is convergent and well defined at $x = \pm a$ because l'Hospital's rule shows

$$\lim_{x \to \pm a} \frac{\cos x - \cos a}{x^2 - a^2} = \lim_{x \to \pm a} -\frac{\sin x}{2x} = -\frac{\sin a}{2a}$$

We evaluate I by Considering

$$J = \oint_C \frac{e^{iz} - \cos a}{z^2 - a^2}\, dz$$

where the Contour C is depicted below



Because there are no poles enclosed by C

$$J = \oint_C \frac{e^{iz} - \cos a}{z^2 - a^2}\, dz$$

$$= \left\{ \int_{-R}^{-a-r_1} + \int_{-a+r_1}^{a-r_2} + \int_{a+r_2}^{R} + \int_{C_{r_1}} + \int_{C_{r_2}} + \int_{C_R} \right\} \frac{e^{iz} - \cos a}{z^2 - a^2}\, dz = 0 \longrightarrow (1)$$

$$\left| \int_{C_R} \frac{e^{iz} - \cos a}{z^2 - a^2}\, dz \right| \le \int_{C_R} \left| \frac{e^{iz} - \cos a}{z^2 - a^2} \right| |dz|$$

$$\le \int_{C_R} \frac{|e^{iz}| + |\cos a|}{|z^2 - a^2|}\, |dz|$$

$\sin\theta \ge \frac{2\theta}{\pi}$

$-\sin\theta \le -\frac{2\theta}{\pi}$

$e^{-R\sin\theta}$

$$\le \int_{C_R} \frac{|e^{iz}| + |\cos a|}{|z|^2 - |a|^2}\, |dz|$$

$z = Re^{i\theta}$

$dz = Rie^{i\theta} d\theta$

$|dz| = R\, d\theta$

$$= \int_{C_R} \frac{|e^{-R\sin\theta}| + |\cos a|}{R^2 - |a|^2}\, R\, d\theta$$

$$\le \int_{C_R} \frac{|e^{-R\frac{2\theta}{\pi}}| + |\cos a|}{R^2 - a^2}\, R\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{|e^{-R2\theta/\pi}|}{R^2-a^2} R\,d\theta + \int_0^{\pi} \frac{|\cos\theta|}{R^2-a^2} R\,d\theta$$

$$= -\frac{2\pi R}{R^2-a^2}\left[e^{-\frac{R2\theta}{\pi}}\right]_0^{\pi/2} + \frac{|\cos\theta|}{R^2-a^2} R\pi$$

$$= \frac{-2\pi R}{R^2-a^2}\left(e^{-R}-1\right) + \frac{|\cos\theta|}{R^2-a^2} R\pi \to 0 \text{ as } R\to\infty$$

$$\lim_{r_1\to 0} \int_{C_{r_1}} \frac{e^{iz}-\cos a}{z^2-a^2} = -i\pi \operatorname{Res}\left(\frac{e^{iz}-\cos a}{z^2-a^2}, -a\right)$$

$$= -i\pi\left(\frac{e^{iz}-\cos a}{2z}\right)_{z=a} = \frac{\pi \sin a}{2a}$$

using in (1) and taking limit when $R\to\infty, r\to 0$

$$\int_{-\infty}^{+\infty} \frac{e^{ix}-\cos a}{x^2-a^2} = -\pi\frac{\sin a}{a}$$

By taking the real part

$$I = -\pi \sin a/a$$

Example # Prove that

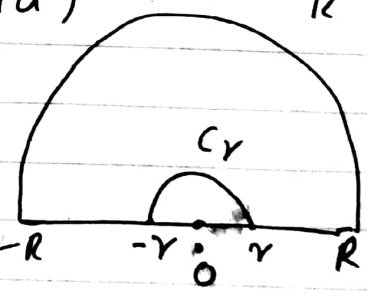
$$\int_0^{\infty} \frac{\sin x\,dx}{x(x^2+a^2)} = \frac{\pi}{2a^2}\left(1-e^{-a}\right) \quad (a>0)$$

Sol Consider $\int_C \frac{e^{iz}\,dz}{z(z^2+a^2)}$

where C is a contour shown in fig

The poles are at $z=0$ $z=\pm ai$ of order 1

The $z=0$ is on real axis and $z=ai$ in upper



half plane

$$R_x(f,0) = \lim_{z\to 0}\left[\frac{e^{iz}}{z(z^2+a^2)}z\right] = \frac{1}{a^2}$$

$$R_p(f,ai) = \lim_{z\to ai}\left[\frac{e^{iz}(z-ai)}{z(z+ai)(z-ai)}\right]$$

$$= \frac{-e^{-a}}{2a^2}$$

$$\int_{-\infty}^{+\infty}\frac{e^{iz}\,dz}{z(z^2+a^2)} = \int_{-\infty}^{+\infty}\frac{e^{ix}}{x(x^2+a^2)} = 2\pi i\,\text{Res}_p(f,ai) + i\pi\,\text{Res}_x(f,0)$$

$$\Rightarrow \int_{-\infty}^{+\infty}\frac{(\sin x + i\cos x)\,dx}{x(x^2+a^2)} = 2\pi i \times -\frac{e^{-a}}{2a^2} + i\pi \times \frac{1}{a^2}$$

Comparing imaginary parts

$$\int_{-\infty}^{+\infty}\frac{\sin x\,dx}{x(x^2+a^2} = \frac{\pi}{a^2} - \frac{\pi}{a^2}e^{-a}$$

$$= \frac{\pi}{a^2}(1-e^{-a})$$

$$2\int_0^{\infty}\frac{\sin x\,dx}{x(x^2+a^2)} = \frac{\pi}{a^2}(1-e^{-a})$$

$$\int_0^{\infty}\frac{\sin x\,dx}{x(x^2+a^2)} = \frac{\pi}{2a^2}(1-e^{-a})$$

Example # prove that

$$\int_0^{\infty}\frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a) \quad b>a>0$$

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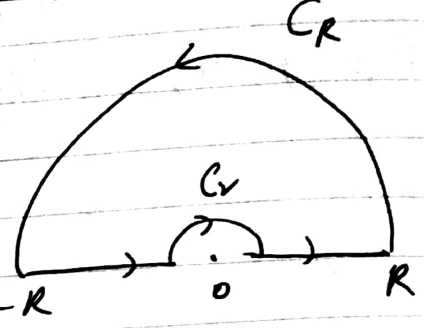
Sol # The Given integral $\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx$

Consider $\int_C \frac{e^{iaz} - e^{ibz}}{z^2}\,dz$



The pole $z=0$ of order 2 lies on the real axis and no pole in the upper half plane

$$R_x(f,0) = \frac{d}{dz}\left[\frac{e^{aiz} - e^{biz}}{z^2}\cdot z^2\right]_{z=0}$$

$$= \left[ai e^{aiz} - bi e^{biz}\right]_{z=0}$$

$$= i(a-b)$$

$$\int_{-\infty}^{+\infty} \frac{e^{aiz} - e^{biz}}{z^2}\,dz = \pi i(a-b)$$

$$\int_{-\infty}^{+\infty} \frac{\cos ax - \cos bx + i(\sin ax + \sin bx)}{x^2}\,dx = -\pi(a-b)$$

Comparing real parts

$$\int_{-\infty}^{+\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = -\pi(a-b) = \pi(b-a)$$

$$2\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \pi(b-a)$$

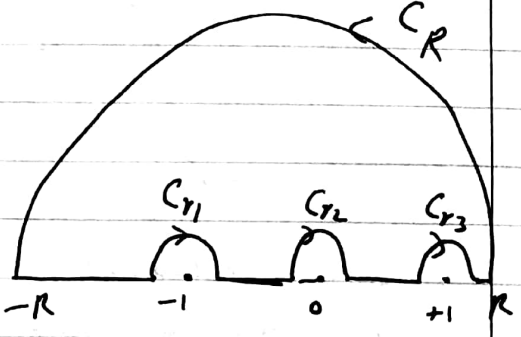
$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a)$$

Example # prove that $\int_0^\infty \frac{\sin \pi x}{x(1-x^2)}\,dx = \pi$

Sol # The Given integral can be written as

$$\int_C \frac{e^{\pi i z}\,dz}{z(1-z^2)}$$

The poles are at $z = 0$ $z = \pm 1$. All lie on real axis and are of order 1



$$R_x(f,0) = \lim_{z\to 0}\left[\frac{e^{\pi i z}\cdot z}{z(1-z^2)}\right]$$

$$= 1$$

$$R_x(f,1) = \lim_{z\to 1}\left[\frac{e^{\pi i z}(z-1)}{z(1-z)(1+z)}\right]$$

$$= \frac{e^{\pi i z}}{-z(1+z)} = \frac{1}{2} \qquad (\because e^{\pi i} = -1)$$

$$R_x(f,-1) = \lim_{z\to -1}\left[\frac{e^{\pi i z}(1+z)}{z(1-z)(1+z)}\right] = \frac{1}{2}.$$

$$\int_{C_{r_1}} \frac{e^{\pi i z}\,dz}{z(1-z^2} = -i\pi R_x(f,-1) = -i\pi \cdot 1/2$$

$$\int_{C_{r_2}} \frac{e^{\pi i z}\,dz}{z(1-z^2)} = -i\pi R_x(f,0) = -i\pi$$

$$\int_{C_{r_3}} \frac{e^{\pi i z}\,dz}{z(1-z^2)} = -i\pi R_x(f,1) = -i\pi \cdot \frac{1}{2}.$$

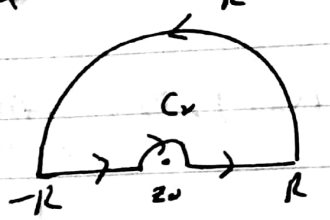
$$\int_{-\infty}^{+\infty} \frac{e^{\pi i z}\,dz}{z(1-z^2)} = 2\pi i$$

$$2\int_0^{\infty} \frac{\sin x\,dx}{x(1-x^2)} = 2\pi$$

$$\int_0^{\infty} \frac{\sin x\,dx}{x(1-x^2)} = \pi$$

Example # Evaluate $\int_{-\infty}^{+\infty} \frac{(x^2+a^2)\sin mx}{x(x^2+b^2)}\,dx$

Sol Consider the integral $\int_{-\infty}^{+\infty} \frac{(z^2+a^2)\,e^{imz}}{z(z^2+b^2)}\,dz$



The poles are at $z=0$, $z=\pm ib$

The pole on real axis is $z=0$ of order 1

The pole in upper half plane is $z=ib$ of order 1

$$R_x(f,0) = \lim_{z\to 0}\left[\frac{(z^2+a^2)\,e^{imz}\cdot z}{(z^2+b^2)\cdot z}\right] = \frac{a^2}{b^2}$$

$$R_x(f,bi) = \lim_{z\to bi}\left[\frac{(z^2+a^2)\,e^{imz}(z-bi)}{z(z+bi)(z-bi)}\right]$$

$$= \frac{(a^2-b^2)\,e^{-mb}}{-2b^2} = \frac{(b^2-a^2)\,e^{-mb}}{2b^2}$$

$$\int_{-\infty}^{+\infty} \frac{(z^2+a^2)\,e^{imz}}{z(z^2+b^2)}\,dx = 2\pi i \times \frac{(b^2-a^2)\,e^{-mb}}{2b^2} + i\pi\cdot\frac{a^2}{b^2}$$

Compairing imaginary parts

$$2\int_{-\infty}^{+\infty} \frac{(x^2+a^2)\sin mx}{x(x^2+b^2)} = \frac{\pi(b^2-a^2)}{b^2}e^{-mb} + \frac{\pi a^2}{b^2}$$

$$= \frac{\pi}{b^2}\left[(b^2-a^2)\,e^{-mb} + a^2\right]$$

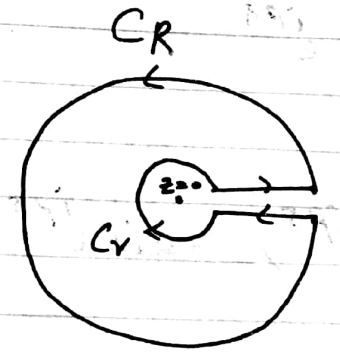
Type:- Integrals With Branch Points

We consider integrands that involve branch points. To evalute the integrals, we introduce branch cuts associated with the relevant multivalued functions:

If $z^{\alpha-1}$ be many-valued function, then an integral of the type

$$\int_0^\infty x^{\alpha-1} f(x)\,dx.$$

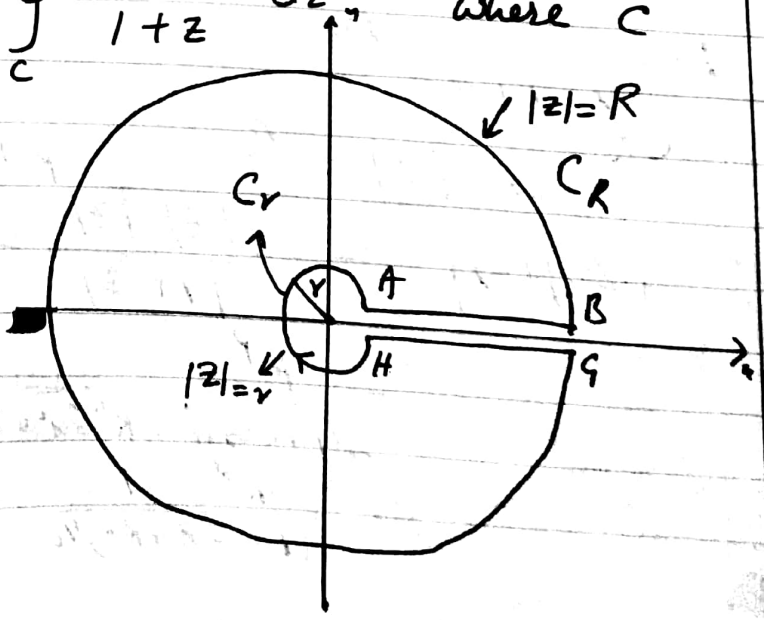
where α is not integer can be evaluated by Contour integration by taking the integral round a Circle $|z|=R$ and a small circle $|z|=r$ which enclosed the branch point $z=0$ and a cross cut along the real axis from 0 to ∞ ie joining the ends of the two circles Such a Contour is also called a keyhole Contour.



Example # Evaluate $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx \qquad 0<p<1.$

Sol #

Consider $\oint_C \frac{z^{p-1}}{1+z}\,dz$, where C is the "keyhole" Contour depicted. Since $z=0$ is a multiple point, therefore it is avoided by Such a Contour. Here AB and GH are



are actually coincident with the x-axis but are shown separated for visual purpose. The f is meromorphic on the contour consisting of

(i) x-axis from r to R.
(ii) The circle $|z|=R$ i.e C_R
(iii) The x-axis from R to r.
(iv) The circle $|z|=r$ i.e C_r

Let $r<1<R$ so that $z=-1=e^{i\pi}$ is a simple pole inside the contour.

$$R(f,-1) = \lim_{\substack{z\to -1 \\ z\to e^{i\pi}}} \frac{z^{p-1}}{z+1}\cdot(z+1) = e^{i(p-1)\pi}$$

$$= e^{ip\pi - i\pi}$$

$$= -e^{ip\pi}$$

By Residue theorem

$$\oint_C \frac{z^{p-1}}{1+z}\,dz = 2\pi i\,(-e^{ip\pi}) = -2\pi i e^{ip\pi}$$

$$\Rightarrow \left(\underbrace{\int_r^R}_{I_1} + \underbrace{\int_{C_R}}_{I_2} + \underbrace{\int_R^r}_{I_3} + \underbrace{\int_{C_r}}_{I_4}\right) \frac{z^{p-1}}{1+z}\,dz = -2\pi i e^{ip\pi} \longrightarrow (1)$$

$$I_2 = \int_{C_R} \frac{z^{p-1}}{1+z}\,dz$$

on $|z|=R$

$$|z f(z)| = \frac{|z|\,|z|^{p-1}}{|1+z|} = \frac{R\cdot R^{p-1}}{|1+R\cos\theta + iR\sin\theta|}$$

$$= \frac{R^p}{\sqrt{(1+R\cos\theta)^2 + R^2\sin^2\theta}} = \frac{R^p}{(1+2R\cos\theta+R^2)^{1/2}}$$

$$\le \frac{R^p}{(1-2R+R^2)^{1/2}} = \frac{R^p}{R-1} \to 0 \text{ as } R\to\infty \quad \because p<1$$

$$|Z f(z)| \underset{OR}{=} \frac{|z|\,|z|^{p-1}}{|z+1|} \leq \frac{|z|\,|z|^{p-1}}{|z|-1|} = \frac{R \cdot R^{p-1}}{R-1} = \frac{R^p}{R-1}$$

$$\to 0 \text{ as } R \to \infty$$

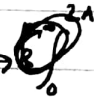
Hence

$$I_2 \to 0 \quad \text{as } R \to \infty$$

$$\underline{OR}$$

$$I_2 = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}\, iRe^{i\theta}\, d\theta}{1+Re^{i\theta}}$$

$$= R^{p-1} i \int_0^{2\pi} \frac{e^{ip\theta}\, d\theta}{1/R + e^{i\theta}} d\theta \to 0 \quad \text{as } R \to \infty$$

becaus $\frac{1}{R} \to 0$ & $I_2 \to$  $\because p<1, \frac{1}{R^{1-p}} \to 0$

Again

$$\lim_{z \to 0} (z-0) f(z) = \lim_{z \to 0} \frac{z \cdot z^{p-1}}{1+z} = 0 = k$$

So

$$\lim_{r \to 0} \int_{C_r} \frac{z^{p-1}}{1+z} = i\,0\,(2\pi) = 0$$

Integral $I_1 = \int_r^R \frac{z^{p-1}}{1+z} dz$

$$= \int_r^R \frac{x^{p-1}}{1+x} dx \qquad z = x \text{ on } AB$$

Integral $I_3 = \int_R^r \frac{z^{p-1}}{1+z} dz$

We use $z = |z| e^{i2\pi} = x e^{i2\pi}$ $\quad R < x < R$

Since the argument of z is increased by 2π in going round the circle C_R or because $z^\alpha = e^{\alpha \log z}$

and after going round the circle C_R, the value of $\log z$ changes by $2\pi i$. Thus

$$I_3 = \int_R^r \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\,dx = e^{2\pi(p-1)i}\int_R^r \frac{x^{p-1}}{1+x}\,dx$$

So ① becomes

$$\int_r^R \frac{x^{p-1}}{1+x}\,dx + \int_{C_R} \frac{z^{p-1}}{1+z}\,dz + \int_R^r \frac{e^{2\pi(p-1)i}\,x^{p-1}}{1+x}\,dx$$

$$+ \int_{C_r} \frac{z^{p-1}}{1+z}\,dz = -2\pi i\, e^{ip\pi}.$$

Hence as $R\to\infty$ and $r\to 0$

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx + 0 + \int_\infty^0 \frac{e^{2\pi(p-1)i}\,x^{p-1}}{1+x}\,dx$$

$$+0 \qquad \infty \qquad = -2\pi i\, e^{ip\pi}$$

$$\left(1 - e^{2\pi(p-1)i}\right)\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = -2\pi i\, e^{ip\pi}$$

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{-2\pi i\, e^{ip\pi}}{1 - e^{2\pi pi - 2\pi i}}$$

$$= \frac{-2\pi i\, e^{ip\pi}}{1 - e^{2\pi pi}}$$

$$= \frac{-2\pi i}{e^{-ip\pi} - e^{ip\pi}} = \frac{2\pi i}{e^{ip\pi} - e^{-ip\pi}}$$

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\left(\frac{e^{ip\pi} - e^{-ip\pi}}{2i}\right)} = \frac{\pi}{\sin \pi p}$$

Note # Above example was the case in which pole lies on the -ve part of the real axis and integral is of the form

$$\int_0^\infty x^{\alpha-1} f(x)\,dx \quad \text{or} \quad \int_0^\infty x^{\alpha} f(x)\,dx \quad \text{where}$$

α is a fraction.

Method #

(i) Replace x by z and $x^{\alpha-1} = z^{\alpha-1} = e^{(\alpha-1)\log z}$

$\because$ pole lies on the -ve part of the real axis

$\therefore$ For z on +ve real axis $\text{Arg}\, z = 0$

$\log z = \log|z| + i0 = \ln|z|$

on +ve part of x-axis $\text{Arg}\, z = \pi$

$z = |z|(\cos\pi + i\sin\pi) = -|z| \Rightarrow |z| = -z$

$\Rightarrow \log z = \log|z| + i\pi = \log(-z) + i\pi$

$\therefore \Phi(z) = e^{(\alpha-1)[\log(-z)+i\pi]} f(z)$

(ii) Calculate the poles of $f(z)$ and consider all the poles which lie in the whole Complex plane. Calculate the Residue at these poles and.

$$\int_0^\infty x^{(\alpha-1)} f(x)\,dx = -\frac{\pi}{\sin\pi\alpha} \cdot e^{-\alpha\pi i} \cdot \Sigma R.$$

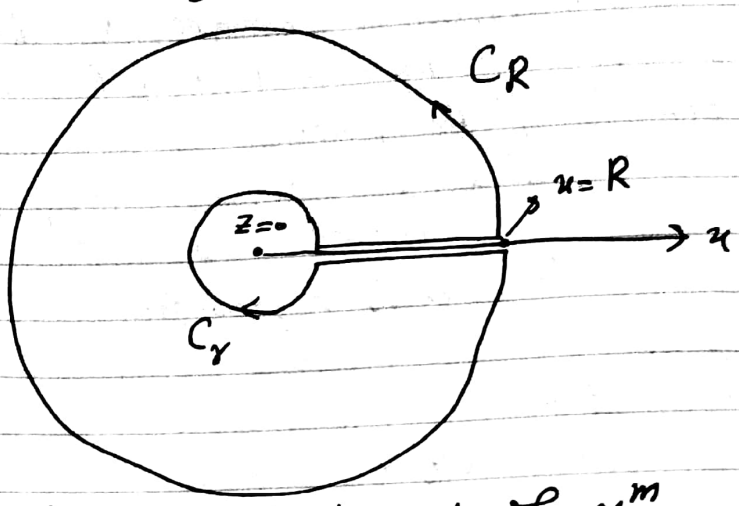
Example # Evaluate $I = \int_0^\infty \frac{x^{m-1}}{x^2+1}\,dx$ $\quad 0 < m < 2$

The condition on m is required for the convergence of the integral.

Sol # Consider the contour integral

$$J = \oint_C \frac{z^{m-1}}{z^2+1}\,dz$$

where C is a 'keyhole' Contour as shown



We take the principal branch of $z^m : z = re^{i\theta}$, $0 \le \theta < 2\pi$.

poles are roots of $z^2 = -1 = e^{i\pi}$

$$z_k = e^{\frac{i(2k\pi+\pi)}{2}} \qquad k = 0, 1$$

$$z_0 = e^{i\pi/2}$$

$$z_1 = e^{i3\pi/2}$$

$$R(f, e^{i\pi/2}) = \left[\frac{z^{m-1}}{2z}\right]_{z=e^{i\pi/2}} = \frac{e^{i(m-1)\frac{\pi}{2}}}{2e^{i\pi/2}}$$

$$= \frac{e^{i(m-1)\pi/2}}{2i}$$

$$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$$

$$R(f, e^{3i\pi/2}) = \left[\frac{z^{m-1}}{2z}\right]_{z=e^{3i\pi/2}}$$

$$= \frac{e^{3(m-1)i\pi/2}}{2e^{3i\pi/2}} = \frac{e^{3i(m-1)\pi/2}}{2(\cos 3\pi/2 + i\sin 3\pi/2)}$$

$$= \frac{e^{\frac{3i(m-1)\pi}{2}}}{-2i}$$

$$\text{Sum of Residues} = \frac{e^{i(m-1)\pi/2}}{2i} - \frac{e^{\frac{3i(m-1)\pi}{2}}}{2i}$$

By Residue Theorem.

$$\oint_C \frac{z^{m-1}}{z^2+1}dz = \int_r^R \frac{x^{m-1}}{x^2+1}dx + \int_R^r \frac{(xe^{2\pi i})^{m-1}}{(xe^{2\pi i})^2+1}dx$$

$$+\left(\int_{C_r} + \int_{C_R}\right)\frac{z^{m-1}}{z^2+1}dz = 2\pi i\left[\frac{e^{i(m-1)\frac{\pi}{2}}}{2i} - \frac{e^{3i(m-1)\frac{\pi}{2}}}{2i}\right] \rightarrow (1)$$

$\because$ $(z-0)f(z) = \frac{z\cdot z^{m-1}}{z^2+1} = \frac{z^m}{z^2+1}$

$\lim_{z\to} zf(z) = 0 = k.$

$\Rightarrow \lim_{r\to 0}\int_{C_r}\frac{z^{m-1}}{z^2+1} = ik(2\pi) = i(0)(2\pi) = 0$

on $|z| = R$

$$|zf(z)| = \left|\frac{z\cdot z^{m-1}}{z^2+1}\right| = \frac{|z^m|}{|z^2+1|}$$

$$\leq \frac{|z|^m}{|z|^2-1} = \frac{R^m}{R^2-1} \to 0 \text{ as } R\to\infty$$

$\because 0 < m < 2$

Hence $\lim_{R\to\infty}\int_{C_R}\frac{z^{m-1}}{1+z^2}dz = 0$

By taking limit $r\to 0$, $R\to\infty$ from (1)

$$\int_0^\infty \frac{x^{m-1}}{1+x^2}dx + \int_\infty^0 \frac{x^{m-1}e^{2(m-1)\pi i}}{x^2+1}dx$$

$e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

$$= 2\pi i\left(\frac{e^{i\frac{\pi}{2}(m-1)}}{2i} - \frac{e^{\frac{3i\pi(m-1)}{2}}}{2i}\right)$$

$$\int_0^\infty \frac{x^{m-1}dx}{1+x^2} - \int_\infty^0 \frac{x^{m-1}\cdot e^{2\pi i m}}{1+x^2}dx$$

$$= 2\pi i\left(\frac{-e^{im\frac{\pi}{2}}}{2i} - \frac{e^{i\frac{3\pi m}{2}}}{2i}\right)$$

$$(1-e^{2\pi mi})\int_0^\infty \frac{x^{m-1}}{1+x^2}dx = -\pi i\left(e^{\frac{2imn\pi}{2}} + e^{\frac{3m\pi i}{2}}\right)$$

Hence

$$I = -\pi i\, e^{\frac{imn\pi}{2}}\left(\frac{1+e^{imn\pi}}{1-e^{2imn\pi}}\right)$$

$$= -\pi i\, e^{\frac{imn\pi}{2}}\left[\frac{1+e^{\frac{imn\pi}{2}}\cdot e^{imn\pi}}{1-e^{imn\pi}\cdot e^{imn\pi}}\right]$$

$$= -\pi i\, e^{\frac{imn\pi}{2}}\left[\frac{e^{-\frac{imn\pi}{2}}+e^{imn\pi/2}}{e^{-\frac{imn\pi}{2}}}\right]\Bigg/\left(\frac{e^{-imn\pi}-e^{imn\pi}}{e^{-imn\pi}}\right)$$

$$= -\pi i\, e^{\frac{imn\pi}{2}}\cdot e^{-imn\pi}\cdot e^{\frac{imn\pi}{2}}\,\frac{e^{imn\pi/2}+e^{-imn\pi/2}}{\left(e^{imn\pi}-e^{-imn\pi}\right)}$$

$$= \pi\,\frac{e^{\frac{imn\pi}{2}}+e^{-\frac{imn\pi}{2}}}{e^{imn\pi}-e^{-imn\pi}}\Big/2$$

$$= \pi\,\frac{\cos\frac{mn\pi}{2}}{\sin mn\pi} = \frac{\pi}{2\sin\left(\frac{mn\pi}{2}\right)} = \frac{\pi}{2}\operatorname{cosec}\left(\frac{mn\pi}{2}\right)$$

Deduction

Deduce that

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}dx = \frac{\pi^3}{8}$$

We have $\int_0^\infty \frac{x^{m-1}}{1+x^2}dx = \frac{\pi}{2}\operatorname{cosec}\left(\frac{mn\pi}{2}\right)$

Diff w.r.t m

$$\int_0^\infty \frac{x^{m-1}\log x}{1+x^2} = \frac{\pi}{2}\left(-\operatorname{Cosec}\frac{m\pi}{2}\cot\frac{m\pi}{2}\cdot\frac{\pi}{2}\right)$$

$$= -\frac{\pi^2}{4}\frac{\cos\frac{m\pi}{2}}{\sin^2\frac{m\pi}{2}}$$

Diff again w.r.t. m

$$\int_0^\infty \frac{x^{m-1}(\log x)^2}{1+x^2} = -\frac{\pi^2}{4}\cdot\frac{-\sin^2\frac{m\pi}{2}\sin\frac{m\pi}{2}\cdot\frac{\pi}{2} - \cos\frac{m\pi}{2}\cdot 2\sin\frac{m\pi}{2}\cos\frac{m\pi}{2}\cdot\frac{\pi}{2}}{\sin^4\frac{m\pi}{2}}$$

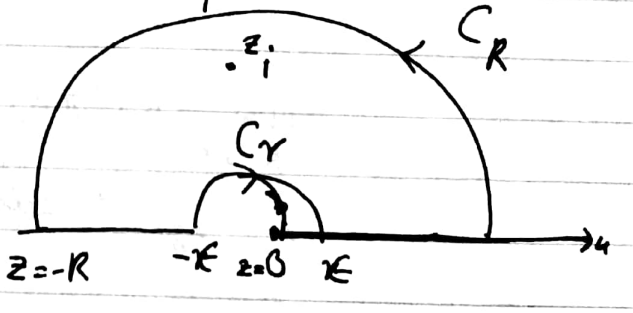
putting $m=1$

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} = -\frac{\pi^2}{4}\cdot\left(-\frac{\pi}{2}\right) = \frac{\pi^3}{8} \quad \text{(Proved)}$$

Example Evaluate $I = \int_0^\infty \frac{\log^2 x}{x^2+1}\,dx$

Sol# Consider $J = \oint_C \frac{\log^2 z}{z^2+1}\,dz$

where C is the contour depicted in fig. and we take the principal branch of $\log z$: $z = re^{i\theta}$, $0 \le \theta \le 2\pi$



poles of $f(z)$ are roots of $z^2+1=0$ and.

$z = e^{i\pi/2}$, $e^{i3\pi/2}$

only $z = i = e^{i\pi/2}$ lie in the upper half plane

$$\operatorname{Res}(f, e^{i\pi/2}) = \left(\frac{\log^2 z}{2z}\right)_{e^{i\pi/2}=i} = \frac{(i\frac{\pi}{2})^2}{2i}$$

$$= i\frac{\pi^2}{8}$$

We have

$$J = \int_R^{\epsilon} \frac{[\log(xe^{i\pi})]^2}{(xe^{i\pi})^2+1} e^{i\pi} dx + \int_{\epsilon}^{R} \frac{\log^2 x}{x^2+1} dx$$

$$+ \left(\int_{C_R} + \int_{C_\epsilon} \right) \frac{\log^2 z}{z^2+1} dz = 2\pi i \left(i\frac{\pi}{8} \right) = -\frac{\pi^3}{4}$$

Consider integral

$$\int_{C_R} \frac{\log^2 z}{z^2+1} dz$$

$$\lim_{|z|\to\infty} z f(z) = \lim_{|z|\to\infty} \frac{z \log^2 z}{z^2+1}$$

$$|z f(z)| = \frac{|z| \, |\log^2 z|}{|z^2+1|}$$

$$\leq \frac{|z| \, |\log|z| + i\,\mathrm{Arg}\, z|^2}{|z|^2 - 1}$$

$$\leq \frac{|z| \left\{ [\log|z|]^2 + (\mathrm{Arg}\, z)^2 \right\}}{|z|^2 - 1}$$

$$\leq \frac{R\left[(\log R)^2 + (\mathrm{Arg}\, z)^2 \right]}{R^2 - 1}$$

$$\lim_{R\to\infty} \frac{R[(\log R)^2 + (\mathrm{Arg}\, z)^2}{R^2-1} \quad \frac{\infty}{\infty}$$

$$= \lim_{R\to\infty} \frac{R \cdot 2(\log R)\cdot\frac{1}{R} + (\log R)^2 + (\mathrm{Arg}\, z)^2}{2R}$$

$$= \lim_{R\to\infty} \frac{2\log R + (\log R)^2 + (\mathrm{Arg}\, z)^2}{R}$$

$$= \lim_{R\to\infty} \frac{2\cdot\frac{1}{R} + 2(\log R)\cdot\frac{1}{R}}{1} = \lim_{R\to\infty} \frac{2+2\log R}{R}$$

$$= \lim_{R\to\infty} \frac{2}{R} = 0$$

$$\Rightarrow z f(z) \to 0 \quad R \to \infty$$

$$\int_{C_R} \frac{\log^2 z}{z^2+1}\,dz \to 0 \qquad \text{as } R\to\infty$$

Also

$$\lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\frac{z\log^2 z}{z^2+1} = \lim_{z\to 0}\frac{\log^2 z}{z+\frac{1}{z}}$$

$$= \lim_{z\to 0}\frac{[\log|z|+i\,\mathrm{Arg}(z)]^2}{z+1/z}$$

$$= \lim_{z\to 0}\frac{0}{1+\frac{1}{z^2}} = 0 = k$$

$$\Rightarrow \lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{\log^2 z}{z^2+1}\,dz = 10(\pi) = 0$$

So

$$\int_{-R}^{\epsilon}\frac{[\log(xe^{i\pi})]^2}{(xe^{i\pi})^2+1}e^{i\pi}\,dx + \int_{\epsilon}^{R}\frac{\log^2 x}{x^2+1}\,dx = -\frac{\pi^3}{4}$$

Taking limit $R\to\infty$ $\epsilon\to 0$ & putting $x=-t$ in 1st integral

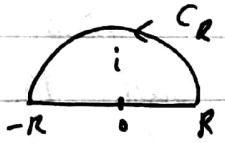
$$\int_{-\infty}^{0}\frac{[\log(xe^{i\pi})]^2 e^{i\pi}}{(xe^{i\pi})^2+1}\,dx + \int_0^{\infty}\frac{\log^2 x}{x^2+1} = -\frac{\pi^3}{4}$$

$$+\int_{\infty}^{0}\frac{(\log x+i\pi)^2}{x^2+1}\,dx + \int_0^{\infty}\frac{\log^2 x}{x^2+1} = -\frac{\pi^3}{4}$$

$$+\int_{-\infty}^{\infty}\frac{\log^2 x-\pi^2+2i\pi\log x}{x^2+1}\,dx + \int_0^{\infty}\frac{\log^2 x}{x^2+1} = -\frac{\pi^3}{4}$$

$$2\int_0^{\infty}\frac{\log^2 x}{x^2+1}\,dx + 2i\pi\int_0^{\infty}\frac{\log x}{x^2+1}\,dx - \pi^2\int_0^{\infty}\frac{dx}{x^2+1} = -\frac{\pi^3}{4}$$

$$\int_0^{\infty}\frac{1}{x^2+1}\,dx = \frac{1}{2}\,2\pi i\left(\frac{1}{2z}\right)_{z=i}$$



$$= \pi/2$$

$$\Rightarrow 2\int_0^{\infty}\frac{\log^2 x}{x^2+1} + 2i\pi\int_0^{\infty}\frac{\log x}{x^2+1}\,dx - \frac{\pi^3}{2} = -\frac{\pi^3}{4}$$

$$2\int_0^\infty \frac{\log^2 x\,dx}{x^2+1} + 2\pi i \int_0^\infty \frac{\log x\,dx}{x^2+1} = \frac{\pi^3}{4}$$

taking the real and imaginary parts

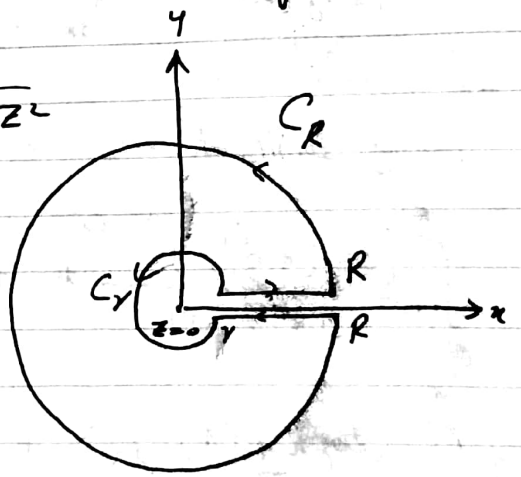
$$I = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^\infty \frac{\log x}{x^2+1}dx = 0$$

Example # If $-1 < p < 1$ and $-\pi < \lambda < \pi$, show by contour integration that

$$\int_0^\infty \frac{x^{-p}\,dx}{1+2x\cos\lambda+x^2} = \frac{\pi}{\sin p\pi}\cdot\frac{\sin p\lambda}{\sin\lambda}$$

Sol Let us consider the integral

$$\int_C \frac{z^{-p}}{1+2z\cos\lambda+z^2}$$



where the Contour C is "keyhole" contour as shown

The poles of $f(z)$ are roots of

$$1+2z\cos\lambda+z^2=0$$

$$(z+e^{i\lambda})(z+e^{-i\lambda}) = 0$$

$$\Rightarrow z = -e^{i\lambda} = e^{i\pi}\cdot e^{i\lambda} = e^{i(\pi+\lambda)}$$

$$z = -e^{-i\lambda} = e^{i(\pi-\lambda)}$$

Amplitudes of the points being $\pi+\lambda$, $\pi-\lambda$

Now $-\pi < \lambda < \pi$

As such the two (poles) amplitudes lie within of z

The poles within the Contour are

$$z = -e^{i\lambda}, -e^{-i\lambda}$$

$$R(f, -e^{i\lambda}) = \lim_{z\to e^{i\lambda}} \frac{(z+e^{i\lambda})\cdot z^{-p}}{(z+e^{i\lambda})(z+e^{-i\lambda})}$$

$$= (-1)^{-p} \frac{e^{-ip\lambda}}{-2i\sin\lambda} \overset{(07)}{=} \frac{e^{-p\pi i} \cdot e^{-ip\lambda}}{-2i\sin\lambda}$$

$$R(f, -e^{-i\lambda}) = \lim_{z \to -e^{-i\lambda}} \frac{z^{-p}}{(z+e^{i\lambda})} = \frac{(-1)^{-p} \cdot e^{ip\lambda}}{-e^{-i\lambda} + e^{i\lambda}}$$

$$= \frac{e^{-p\pi i} \cdot e^{ip\lambda}}{2i\sin\lambda}$$

Sum of the Residues

$$= \frac{e^{-p\pi i} \cdot e^{-ip\lambda}}{-2i\sin\lambda} + \frac{e^{-p\pi i} \cdot e^{ip\lambda}}{2i\sin\lambda}$$

$$= \frac{e^{-p\pi i} \left(e^{ip\lambda} - e^{-ip\lambda}\right)}{2i\sin\lambda}$$

$$= \frac{e^{-p\pi i} \left(2i\sin p\lambda\right)}{2i\sin\lambda} = \frac{e^{-p\pi i}}{\sin\lambda} \cdot \sin p\lambda$$

$$\int_{\gamma}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz - \int_{\gamma}^{R} f(xe^{2\pi i}) d(xe^{2\pi i}) - \int_{C_\gamma} f(z)\,dz.$$

$$= 2\pi i \,(\text{Sum of the Residues}) \longrightarrow (1)$$

Here

$$\left| \int_{C_R} f(z)\,dz \right| \leq \int_{C_R} \frac{|z|^{-p}\,|dz|}{|z+e^{i\lambda}|\,|z+e^{-i\lambda}|}$$

$$\leq \frac{R^{-p} \cdot 2\pi R}{\left[|z| - |e^{i\lambda}|\right]\left[|z| - |e^{-i\lambda}|\right]}$$

$$\leq \frac{R^{-p} \cdot 2\pi R}{(R-1)(R-1)} = \frac{2\pi}{R^{1+p}\left(1-\frac{1}{R}\right)^2}$$

$\to 0$ as $R \to \infty$ provided $1+p > 0$, $p > -1$

Similarly $\left| \int_{C_\gamma} f(z)\,dz \right| \leq \frac{2\pi\, \gamma^{1-p}}{(1+\gamma)^2} \to 0$ as $z \to 0$ or $\gamma \to 0$

with these results as $R \to \infty$ and $\gamma \to 0$ (1) reduces to

$$\int_0^\infty f(x)\,dx - \int_0^\infty f(xe^{2\pi i})\,dx = 2\pi i \cdot e^{-p\pi i}\cdot \frac{\sin p\lambda}{\sin \lambda}$$

$$\int_0^\infty \frac{x^{-p}\,dx}{1+2x\cos\lambda + x^2} - \int_0^\infty \frac{(xe^{2\pi i})^{-p}\cdot e^{2\pi i}\,dx}{1+2xe^{2\pi i}\cos\lambda + (xe^{2\pi i})^2}$$

$$= 2\pi i e^{-p\pi i}\cdot \frac{\sin p\lambda}{\sin\lambda}$$

$e^{2\pi i} = 1$

$$(1-e^{-2p\pi i})\int_0^\infty \frac{x^{-p}\,dx}{1+2x\cos\lambda + x^2} = 2\pi i\, e^{-p\pi i}\frac{\sin p\lambda}{\sin\lambda}$$

$$\int_0^\infty \frac{x^{-p}\,dx}{1+2x\cos\lambda + x^2} = \frac{2\pi i\cdot e^{-p\pi i}\sin p\lambda}{e^{-p\pi i}\left(e^{p\pi i}-e^{-p\pi i}\right)\sin\lambda}$$

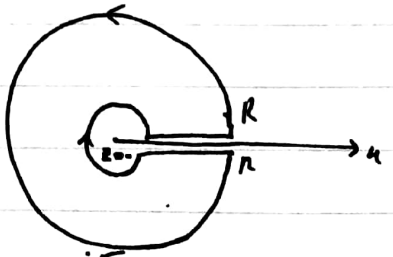
$$= \frac{\pi}{\left(\frac{e^{p\pi i}-e^{-p\pi i}}{2i}\right)}\cdot\frac{\sin p\lambda}{\sin\lambda}$$

$$= \frac{\pi}{\sin p\pi}\cdot\frac{\sin p\lambda}{\sin\lambda}$$

Example# prove that

$$\int_0^\infty \frac{x^{1/4}}{x^2+x+1}\,dx = \pi\left(1-\frac{1}{\sqrt{3}}\right)$$

Sol Consider the integral $\oint_C \frac{z^{1/4}}{z^2+z+1}\,dz$ and 'keyhole' contour as shown.



here +ve x-axis is branch cut and zero is also excluded

poles are solution of

$$z^2+z+1 = 0$$

$$z = \frac{-1\pm i\sqrt{3}}{2} = -\frac{1}{2}+i\frac{\sqrt{3}}{2},\ -\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$R\left(f, \frac{-1+i\sqrt{3}}{2}\right) = \left(\frac{z^{1/4}}{2z+1}\right)_{z=-\frac{1}{2}+i\frac{\sqrt{3}}{2}} = \frac{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{1/4}}{i\sqrt{3}}$$

$$= \frac{\left(e^{\frac{2\pi}{3}i}\right)^{1/4}}{i\sqrt{3}} = \frac{e^{\pi/6\,i}}{i\sqrt{3}}$$

$$R\left(f, \frac{-1-i\sqrt{3}}{2}\right) = \frac{\left(e^{\frac{-2\pi i}{3}}\right)^{1/4}}{-i\sqrt{3}} = \frac{e^{\frac{-\pi}{6}i}}{-i\sqrt{3}}$$

$$\text{Sum of Residues} = \frac{e^{\frac{\pi}{6}i} - e^{\frac{-\pi}{6}i}}{i\sqrt{3}} = \frac{2i\sin\frac{\pi}{6}}{i\sqrt{3}}$$

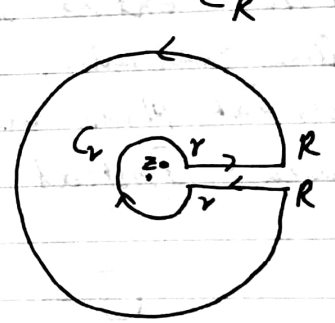
$$=$$

Example prove that $\int_0^\infty \frac{\sqrt{x}\,dx}{(1+x)^2} = \frac{\pi}{2}$

Sol Consider the integral

$$\int_C \frac{\sqrt{z}\,dz}{(1+z)^2}$$

and the (key hole) contour C as shown



poles are Roots of

$$(1+z)^2 = 0$$

$$z = -1$$

order of pole is 2

$$R(f, -1) = \lim_{z\to -1} \frac{d}{dz}(z+1)^2 \frac{\sqrt{z}}{(z+1)^2}$$

$$= \lim_{z\to -1} \frac{1}{2\sqrt{z}} \qquad -1 = e^{\pi i}$$

$$= \frac{1}{2\sqrt{e^{\pi i}}} = \frac{1}{2e^{\frac{\pi}{2}i}} = \frac{1}{2i}$$

By Residue Theorem

$$\int_r^R \frac{\sqrt{x}}{(1+x)^2}dx + \int_{C_R} \frac{\sqrt{z}\,dz}{(1+z)^2} - \int_r^R \frac{\sqrt{xe^{2\pi i}}}{(1+xe^{2\pi i})^2} dx e^{2\pi i}$$

$$+ \int_{C_r} \frac{\sqrt{z}}{(1+z)^2}dz = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

Taking limit $\gamma \to 0$, $R \to \infty$

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx + \int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx = 2\pi$$

$$2\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx = 2\pi$$

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx = \pi$$

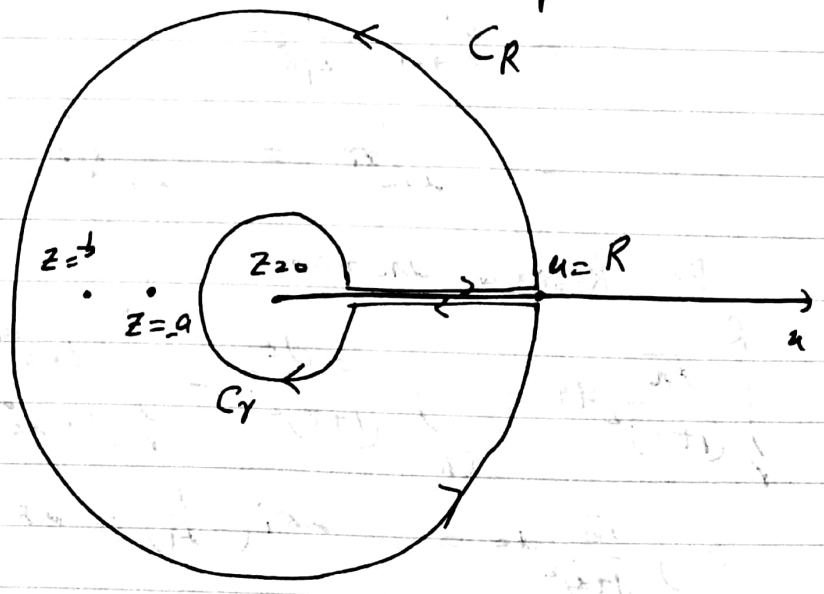
Example # Use Contour Integration to evaluate

$$I = \int_0^\infty \frac{dx}{(x+a)(x+b)} \qquad a, b > 0$$

Sol # Because the integrand is not even, we can not extend our integration region to the entire real line. Hence our previous method for upper half plane and real line from $-R$ to R can not be used. Instead we consider the contour integral

$$J = \oint_C \frac{\log z\, dz}{(z+a)(z+b)}$$

where C is the "keyhole" Contour depicted.



We take $\log z$ to be on its principal branch $z = r e^{i\theta} : 0 \le \theta < 2\pi$ and choose a branch cut along the x-axis, $0 \le x < \infty$

An essential ingredient of the method is that owing to the location of the branch cut, the sum of the integrals on each side of the cut do not cancel.

$$J = \oint_C \frac{\log z \, dz}{(z+a)(z+b)}$$

$$= \int_r^R \frac{\log x \, dx}{(x+a)(x+b)} + \int_R^r \frac{\log(x e^{2\pi i}) \, dx}{(x+a)(x+b)}$$

$$+ \left(\int_{C_r} + \int_{C_R} \right) \frac{\log z \, dz}{(z+a)(z+b)}$$

$$= 2\pi i \{ \text{Sum of Residues at poles } z = -a, \; z = -b \}$$

$$= 2\pi i \left\{ \left(\frac{\log z}{z+a} \right)_{z=-b} + \left(\frac{\log z}{z+b} \right)_{z=-a} \right\}$$

$$=$$ Taking limit when $r \to 0, R \to \infty$

$$\lim_{r \to 0} \int_{C_r} \frac{\log z \, dz}{(z+a)(z+b)} = 0$$

$$\lim_{R \to \infty} \int_{C_R} \frac{\log z \, dz}{(z+a)(z+b)} = 0$$

using $\log(x e^{2\pi i}) = \log x + 2\pi i$

$\log(-a) = \log|a| + i\pi$ for $a > 0$

$$-2\pi i \int_0^\infty \frac{dx}{(x+a)(x+b)} = 2\pi i \left(\frac{\log b + i\pi}{a-b} + \frac{\log a + i\pi}{b-a} \right)$$

$$= 2\pi i \left(\frac{\log b - \log a}{a-b} \right)$$

$$\int_0^\infty \frac{dx}{(x+a)(x+b)} = \frac{\log(b/a)}{b-a}$$

Note # Of Course, we could have evaluated this integral by elementary methods because (we worked this example only for illustrative purposes)

$$I = \frac{1}{b-a}\int_0^\infty \left(\frac{1}{x+a} - \frac{1}{x+b}\right) dx \quad \text{partial fraction.}$$

$$= \frac{1}{b-a}\left[\ln\left(\frac{x+a}{x+b}\right)\right]_0^\infty = \frac{\log(b/a)}{b-a}$$

Case-II When the pole lie on the part of the real axis form of the integral

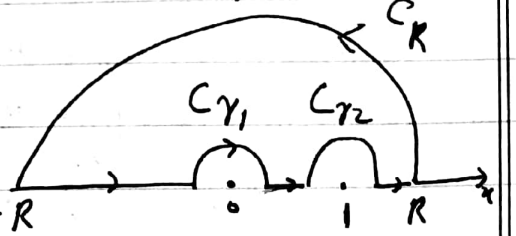
$$\int_0^\infty x^{\alpha-1} f(x)\,dx \quad \text{or} \quad \int_0^\infty x^{\alpha} f(x)\,dx.$$

Example # prove that

$$\int_0^\infty \frac{x^{\alpha-1}}{1-x}\,dx = \pi \cot \alpha\pi \quad , 0 < \alpha < 1$$

Sol # Let us Consider the integral

$$\int_C f(z)\,dz \qquad \text{where } f(z) = \frac{z^{\alpha-1}}{1-z}$$



Choosing the Contour C Consisting of large semi-circle $|z| = R$ indented at $z=0$, $z=1$. Let the radii of indentation be γ_1 & γ_2 respectively

Since there are no poles within the Contour, we have

$$\int_{r_1}^{1-r_2} \frac{x^{\alpha-1}}{1-x}dx + \int_{1+r_2}^{R} \frac{x^{\alpha-1}}{1-x}dx + \int_{R}^{r_1} \frac{(xe^{\pi i})^{\alpha-1}}{1-xe^{\pi i}} d(xe^{\pi i})$$

$$+ \int_{C_R} f(z)dz + \int_{C_{r_1}} f(z)dz + \int_{C_{r_2}} f(z)dz = 0 \quad \rightarrow (1)$$

Now residue of $f(z)$ at $z=0$ is $= \lim_{z\to 0} z \cdot \frac{z^{\alpha-1}}{1-z} = 0 \quad \alpha > 0$

$$\lim_{r_1 \to 0} \int_{C_{r_1}} f(z)dz = i \cdot \text{angle (Residue at } z=0)$$

$$= 0$$

Residue of $f(z)$ at $z=1 = \lim_{z\to 1} (z-1)\frac{z^{\alpha-1}}{1-z}$

$$= \lim_{z\to 1} -z^{\alpha-1} = -1$$

So $$\lim_{r_2 \to 0} \int_{C_{r_2}} f(z)dz = -i(\pi - 0)(-1) = \pi i$$

and $$\left| \int_{C_R} \frac{z^{\alpha-1}}{1-z} dz \right| \leq \int_{C_R} \frac{|z|^{\alpha-1}}{|z-1|} |dz|$$

$$\leq \frac{R^{\alpha-1}}{R-1} \cdot \pi R = \frac{\pi R^{\alpha}}{R-1}$$

$$\rightarrow 0 \text{ as } R \to \infty$$

$$\therefore \alpha < 1$$

with these results as $R \to \infty$ & $r_1, r_2 \to 0$ (1) reduces to

$$\int_{\infty}^{1} \frac{x^{\alpha-1}}{1-x}dx + \int_{0}^{\infty} \frac{x^{\alpha-1}}{1-x}dx + \int_{\infty}^{0} \frac{e^{\alpha\pi i} \cdot x^{\alpha-1}}{1+x}dx + \pi i = 0$$

or $$\int_{0}^{\infty} \frac{x^{\alpha-1}}{1-x}dx - e^{\alpha\pi i}\int_{0}^{\infty} \frac{x^{\alpha-1}}{1+x}dx + \pi i = 0$$

$$\int_0^\infty \frac{x^{\alpha-1}}{1-x}dx - (\cos\alpha\pi + i\sin\alpha\pi)\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx$$

$$= -\pi i$$

Equating real and imaginary parts we find

$$\int_0^\infty \frac{x^{\alpha-1}}{1-x}dx = \cos\alpha\pi \int_0^\infty \frac{x^{\alpha-1}}{1+x}dx \longrightarrow (2)$$

& $\sin\alpha\pi \int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \pi$

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx = \frac{\pi}{\sin\alpha\pi}$$

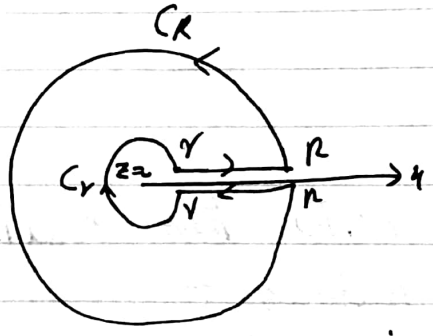
putting in (2)

$$\int_0^\infty \frac{x^{\alpha-1}}{1-x}dx = \cos\alpha\pi \cdot \frac{\pi}{\sin\alpha\pi}$$

$$= \pi\cot\alpha\pi$$

Example # Prove that $\int_0^\infty \frac{\sqrt{x}\log x}{(1+x)^2}dx = \pi$

Sol Consider the integral $\oint_C \frac{\sqrt{z}\log z}{(1+z)^2}dz$ where C is (keyhole) contour.



pole of $f(z)$ is at $z=-1$ of order 2

$$R(f,-1) = \lim_{z\to -1}\frac{d}{dz}\left[\sqrt{z}\log z\right]$$

$$= \lim_{z\to -1}\left[\frac{1}{\sqrt{z}} + \frac{\log z}{2\sqrt{z}}\right] = \left[\frac{1}{i} + \frac{\log e^{\pi i}}{2i}\right]$$

$$= \left[\frac{1}{i} + \frac{\pi}{2}\right]$$

By Residue Theorem

$$\oint_C f(z)dz = 2\pi i\left[\frac{1}{i} + \frac{\pi}{2}\right]$$

$$\int_r^R \frac{\sqrt{x}\log x}{(1+x)^2}dx - \int_1^R \frac{\sqrt{xe^{2\pi i}}\log(xe^{2\pi i})}{(1+xe^{2\pi i})^2}e^{2\pi i}dx + \int_{C_R} f(z)dz + \int_{C_r} f(z)dz$$

$$= 2\pi i\left[\frac{1}{i} + \frac{\pi}{2}\right] = 2\pi + \pi^2 i$$

Taking limit when $r \to 0$ $R \to \infty$

$$\int_0^\infty \frac{\sqrt{x}\log x\,dx}{(1+x)^2} - \int_0^\infty \frac{\sqrt{x}\,e^{\pi i}[\log x + 2\pi i]}{(1+x)^2}dx$$

$$= 2\pi + \pi^2 i$$

$$2\int_0^\infty \frac{\sqrt{x}\log x\,dx}{(1+x)^2} + 2\pi i\int_0^\infty \frac{\sqrt{x}\,dx}{(1+x)^2} = 2\pi + \pi^2 i$$

Comparing real & imaginary parts

$$\int_0^\infty \frac{\sqrt{x}\log x\,dx}{(1+x)^2} = \pi \quad \& \quad \int_0^\infty \frac{\sqrt{x}\,dx}{(1+x)^2} = \frac{\pi}{2}.$$

Type

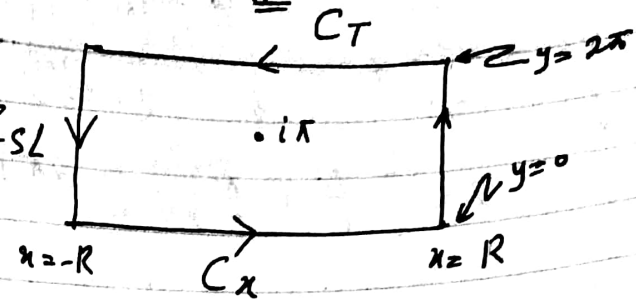
Integrals By Using Rectangular And Other Contours

Example # Evaluate $I = \int_{-\infty}^{+\infty} \frac{e^{px}dx}{1+e^x}$

for $0 < \mathrm{Re}\,p < 1$. The condition on p is required for convergence of the integral.

Sol Consider the contour depicted.

2.1.16



$$f(z) = \frac{e^{pz}}{1+e^z}$$

The poles of $f(z)$ are roots of $e^z = -1 = e^{(\pi + 2n\pi)}$ or

$n = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$z = i(\pi + 2n\pi)$$

The Contour is chosen such that $z = x + iy$, $0 \le y \le 2\pi$

hence the only pole inside the contour is $z = i\pi$

$$\text{Res}\left(\frac{e^{pz}}{1+e^z}; i\pi\right) = = \left(\frac{e^{pz}}{e^z}\right)_{z = i\pi} = e^{(p-1)i\pi}$$

By Cauchy's Residue Theorem

$$\oint_C \frac{e^{pz}\,dz}{1+e^z} = \left(\int_{C_x} + \int_{C_{SR}} + \int_{C_{SL}} + \int_{C_T}\right) \frac{e^{pz}}{1+e^z}\,dz$$

$$= 2\pi i \cdot e^{(p-1)i\pi} \longrightarrow (A)$$

Integral on C_{SR} : $z = R + iy$, $0 \le y \le 2\pi$

$$\left| \int_{C_{SR}} \frac{e^{pz}\,dz}{1+e^z} \right| = \left| \int_0^{2\pi} \frac{e^{p(R+iy)}}{1+e^{R+iy}}\, i\,dy \right|$$

$$\le \frac{e^{pR}}{e^R - 1}\, 2\pi \longrightarrow 0 \text{ as } R \to \infty \quad (\text{Re}\, p < 1)$$

Integral on C_{SL} : $z = -R + iy$, $0 \le y \le 2\pi$

$$\left| \int_{C_{SL}} \frac{e^{pz}}{1+e^z}\,dz \right| = \left| \int_{2\pi}^{0} \frac{e^{p(-R+iy)}}{1-e^{-R+iy}} \right| \le \frac{e^{-pR}}{1-e^{-R}}\, 2\pi \to 0$$

as $R \to \infty$ $(\text{Re}\, p > 0)$

Integral along C_I

on the top $z = x + 2\pi i$, $e^z = e^x$, so

$$\int_{C_I} \frac{e^{pz}}{1+e^z}\,dz = e^{2\pi i p} \int_{+R}^{-R} \frac{e^{px}}{1+e^x}\,dx.$$

$$= -e^{2\pi i p} \int_{-R}^{R} \frac{e^{px}}{1+e^x}\,dx$$

Integral along C_R

along C_R $\quad z = x + 0i = x \quad e^{pz} = e^{px}$

$$\int_{C_R} \frac{e^{px}}{1+e^x}\,dx = \int_{-R}^{R} \frac{e^{px}}{1+e^x}\,dx.$$

Hence putting all of this together and taking limit when $R \to \infty$

$$\int_{-\infty}^{+\infty} \frac{e^{px}\,dx}{1+e^x} \left(1 - e^{2\pi i p}\right) = 2\pi i\, e^{(p-1)i\pi}$$

$$\int_{-\infty}^{+\infty} \frac{e^{px}}{1+e^x}\,dx = \frac{2\pi i\, e^{(p-1)i\pi}}{1-e^{2\pi i p}}$$

$$= 2\pi i \, \frac{e^{-i\pi}}{e^{-ip\pi} - e^{ip\pi}}$$

$$= \frac{-1 \cdot \pi}{\left(\frac{e^{-ip\pi} - e^{ip\pi}}{2i}\right)}$$

$$= \frac{\pi}{\left(\frac{e^{ip\pi} - e^{-ip\pi}}{2i}\right)} = \frac{\pi}{\sin p\pi}$$

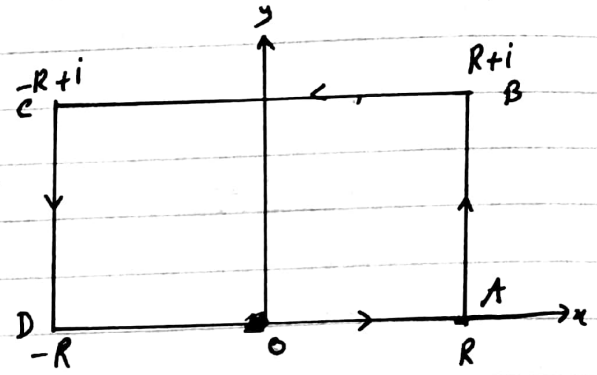
Example # Using the Rectangular Contour Prove that

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x} = \frac{1}{2}\sec\left(\frac{a}{2}\right)$$

provided $|a| < \pi$ or $-\pi < a < \pi$

Sol # Consider

$$\int_C f(z)\,dz = \int_C \frac{e^{az}}{\cosh \pi z}\,dz.$$



poles of $f(z)$ are given by

$$\cosh \pi z = 0$$

$$\frac{e^{\pi z} + e^{-\pi z}}{2} = 0$$

$$e^{\pi z} = -e^{-\pi z} = -\frac{1}{e^{\pi z}}$$

$$e^{2\pi z} = -1 = e^{i(\pi + 2n\pi)} = e^{i(2n+1)\pi}$$

$$\Rightarrow z = \frac{2n+1}{2} i \quad \text{where } n = 0, \pm 1, \pm 2$$

So $f(z)$ has simple poles at $\pm \frac{1}{2}i, \pm \frac{3}{2}i \ldots$ only the simple pole $z = \frac{i}{2}$ lies within the Contour

Residue of $f(z)$ at $z = \frac{i}{2}$

$$= \left[\frac{e^{az}}{\frac{d}{dz}\cosh \pi z}\right]_{z = i/2}$$

$$= \frac{e^{ia/2}}{\pi \sinh \frac{\pi i}{2}} = \frac{e^{ia/2}}{i\pi \sin \pi/2}$$

$$= \frac{e^{ia/2}}{i\pi}$$

By Cauchy Residue Theorem

$$\int_{-R}^{R} f(x)\,dx + \int_0^1 f(R+iy)\,i\,dy + \int_R^{-R} f(x+i)\,dx + \int_1^0 f(-R+iy)\,i\,dy$$

$$= 2\pi i \frac{e^{ia/2}}{i\pi} = 2e^{ai/2} \quad \rightarrow (A)$$

Integral Along AB & CD

$$\left|\int_0^1 f(R+iy)\,i\,dy\right| \le \int_0^1 |f(R+iy)||i||dy|$$

$$\le \int_0^1 \left|\frac{e^{a(R+iy)}}{\cosh\pi(R+iy)}\right| dy \qquad |i|=1$$

$$\le \int_0^1 \frac{2e^{aR}\cdot|e^{diy}|\,dy}{\left|e^{\pi(R+iy)}+e^{-\pi(R+iy)}\right|}$$

$$\le \int_0^1 \frac{2e^{aR}\,dy}{|e^{\pi(R+iy)}|-|e^{-\pi(R+iy)}|}$$

$$\le \frac{2e^{aR}}{e^{\pi R}-e^{-\pi R}} \to 0 \quad \text{as } R\to\infty \quad \text{if } |a|<\pi$$

$$\therefore \int_0^1 f(R+iy)\,i\,dy \to 0 \quad \text{as } R\to\infty$$

Similarly integral along CD tends to zero as $R\to\infty$

Along DA

$$\text{Integral} = \int_{-R}^{R} \frac{e^{ax}}{\cosh\pi x}\,dx.$$

Along BC

$$\text{Integral} = \int_{R}^{-R} \frac{e^{a(x+i)}}{\cosh\pi(x+i)}\,dx$$

$$= -\int_{-R}^{R} \frac{e^{a(x+i)}}{\cosh\pi(x+i)}$$

But $\cosh\pi(x+i) = \cosh i(x+i)\pi = \cos(\pi xi-\pi)$

$$= -\cos\pi xi = -\cosh\pi x$$

$$\text{Integral} = -\int_{-R}^{R} \frac{e^{ax}\cdot e^{ia}}{-\cosh\pi x} = -\int_{R}^{-R} \frac{e^{ax}\cdot e^{ia}}{\cosh\pi x}$$

using these in A and taking limit when $R \to \infty$

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{\cosh \pi x}\,dx - \int_{+\infty}^{-\infty} \frac{e^{ax}\cdot e^{ia}}{\cosh \pi x}\,dx = 2e^{ia/2}$$

$$\Rightarrow (1+e^{ia}) \int_{-\infty}^{+\infty} \frac{e^{ax}\,dx}{\cosh \pi x} = 2e^{ia/2}$$

$$\int_{-\infty}^{+\infty} \frac{e^{ax}\,dx}{\cosh \pi x} = \frac{2e^{ia/2}}{1+e^{ia}}$$

$$= \frac{2}{e^{ia/2}+e^{-ia/2}}$$

$$= \frac{1}{\cos a/2} = \sec a/2$$

$$\int_{-\infty}^{0} \frac{e^{ax}\,dx}{\cosh \pi x} + \int_{0}^{\infty} \frac{e^{ax}}{\cosh \pi x}\,dx = \sec(a/2)$$

Now $\int_{-\infty}^{0} \frac{e^{ax}}{\cosh \pi x}\,dx$

let $x = -t \Rightarrow dx = -dt$

when $x = -\infty \quad t = \infty$

$x = 0 \quad t = 0$

$$I = \int_{+\infty}^{0} \frac{e^{-at}}{\cosh \pi t}(-dt) = \int_{0}^{\infty} \frac{e^{-ax}}{\cosh \pi x}\,dx.$$

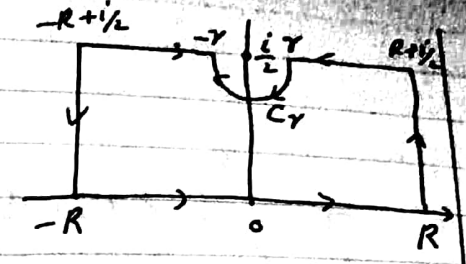
$$\Rightarrow 2\int_{0}^{\infty} \frac{e^{ax}+e^{-ax}}{\cosh \pi x}\,dx = \sec(a/2)$$

$$2\int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x}\,dx = \sec(a/2)$$

$$\int_{0}^{\infty} \frac{\cosh ax\,dx}{\cosh \pi x} = \frac{1}{2}\sec(a/2)$$

Alternative

Consider $\int_C f(z)\,dz = \int_C \frac{e^{az}}{\cosh \pi z}\,dz$



taken round the rectangular contour with vertices at

R, $R+i/2$, $-R+i/2$ and $-R$

indented at $z = i/2$. Since $z = i/2$ is a pole of $f(z)$ clearly $f(z)$ is regular within and on the contour, so that Cauchy's theorem gives

$$\int_{-R}^{R} f(x)\,dx + \int_0^{1/2} f(R+iy)\,i\,dy + \int_R^{r} f(x+\tfrac{1}{2}i)\,dx + \int_{C_r} f(z)\,dz + \int_{-r}^{-R} f(x+\tfrac{1}{2}i)\,dx + \int_{1/2}^{0} f(-R+\tfrac{1}{2}i)\,i\,dy = 0 \quad \longrightarrow (1)$$

But $\lim_{z \to i/2} (z - \tfrac{1}{2}i) f(z) = \lim_{z \to \frac{1}{2}i} \frac{(z-\frac{1}{2}) e^{az}}{\cosh \pi z}$ (2)

$$= \lim_{z \to \frac{1}{2}i} \frac{e^{az} + (z-\frac{1}{2}) e^{az} \cdot a}{\pi \sinh \pi z}$$

$$= \frac{e^{\frac{ai}{2}}}{\pi \sinh \frac{i\pi}{2}} = \frac{e^{\frac{ai}{2}}}{\pi i}$$

$$\therefore \lim_{r \to 0} \int_{C_r} f(z)\,dz = -\pi i \cdot \frac{e^{\frac{ai}{2}}}{\pi i} = -e^{\frac{ai}{2}}$$

$$\left| \int_0^{1/2} f(R+iy)\,i\,dy \right| \le \int_0^{1/2} \left| \frac{e^{a(R+iy)}}{\cosh \pi (R+iy)} \right| |i|\,|dy|$$

$$\le \int_0^{1/2} \frac{2e^{aR}}{|e^{\pi(R+iy)} + e^{-\pi(R+iy)}|}\,dy$$

$$\le \int_0^{1/2} \frac{2e^{aR}}{|e^{\pi(R+iy)}| - |e^{-\pi(R+iy)}|}\,dy$$

$$\leq \frac{e^{aR}}{e^{\pi R} - e^{-\pi R}} \to 0 \quad \text{as } R \to \infty \text{ if } |a| < \pi$$

$$\Rightarrow \int_0^{1/2} f(R+iy)\, i\,dy \to 0$$

Similarly $\int_{1/2}^{0} f(-R+iy)\, i\,dy \to 0$

using these in ① & taking limit when $R \to \infty$, $r \to 0$

$$\int_{-\infty}^{+\infty} f(x)\,dx + \int_{\infty}^{0} f\left(x+\tfrac{1}{2}i\right)dx - e^{\frac{ai}{2}} + \int_{0}^{-\infty} f\left(x+\tfrac{1}{2}i\right)dx = 0$$

$$\int_{-\infty}^{+\infty} f(x)\,dx - \int_{-\infty}^{+\infty} f\left(x+\tfrac{1}{2}i\right)dx = e^{\frac{ai}{2}}$$

$$\int_{-\infty}^{+\infty} \left[\frac{e^{ax}}{\cosh \pi x} - \frac{e^{a(x+\frac{1}{2}i)}}{\cosh \pi (x+\frac{1}{2}i)}\right] dx = e^{\frac{ai}{2}}$$

$$\int_{-\infty}^{+\infty} \left[\frac{e^{ax}}{\cosh \pi x} - \frac{e^{ax} \cdot e^{\frac{ai}{2}}}{\cos i\pi (x+\frac{i}{2})}\right] dx = e^{\frac{ai}{2}}$$

$$\int_{-\infty}^{+\infty} \left[\frac{e^{ax}}{\cosh \pi x} - \frac{e^{ax} \cdot e^{\frac{ai}{2}}}{i \sinh \pi x}\right] dx = e^{\frac{ai}{2}}$$

$$\int_{-\infty}^{+\infty} \left[\frac{e^{ax}}{\cosh \pi x} + \frac{i\, e^{ax}\left(\cos \frac{1}{2}a + i \sin \frac{1}{2}a\right)}{\sinh \pi x}\right] dx$$

$$= \cos \tfrac{1}{2}a + i \sin \tfrac{1}{2}a$$

Equating real and imaginary parts, we get

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{\cosh \pi x} - \int_{-\infty}^{+\infty} \frac{e^{ax} \sin \frac{1}{2}a}{\sinh \pi x}\, dx = \cos \tfrac{1}{2}a \quad \to (2)$$

& $\int_{-\infty}^{+\infty} \frac{e^{ax} \cos \frac{a}{2}}{\sinh \pi x}\, dx = \sin \frac{a}{2}$

$$\text{or} \int_{-\infty}^{+\infty} \frac{e^{ax}}{\sinh \pi x} = \tan a/2 \quad \rightarrow (3)$$

putting (2) in (3)

$$\int_{-\infty}^{+\infty} \frac{e^{ax}\,dx}{\cosh \pi x} - \sin a/2 \cdot \tan a/2 = \cos a/2$$

$$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh \pi x}\,dx + \int_{0}^{\infty} \frac{e^{ax}}{\cosh \pi x}\,dx = \cos a/2 + \sin a/2 \tan a/2$$
$$= \sec a/2$$

$$\int_{0}^{\infty} \frac{e^{-ax}}{\cosh \pi x} + \int_{0}^{\infty} \frac{e^{ax}\,dx}{\cosh \pi x} = \sec a/2$$

$$\int_{0}^{\infty} \frac{e^{ax} + e^{-ax}}{\cosh \pi x}\,dx = \sec(a/2)$$

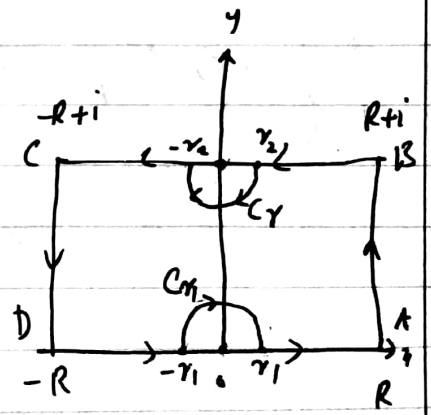
$$\int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x} = \sec(a/2)$$

$$\int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x} = \frac{1}{2}\sec(a/2)$$

Example # $\int_{0}^{\infty} \frac{\sinh ax}{\cosh \pi x}\,dx = \frac{1}{2}\tan\frac{a}{2} \qquad 0 < a < \pi$

Sol # Consider

$$\int_C f(z)\,dz = \int_C \frac{e^{az}\,dz}{\sinh \pi z}$$



poles of $f(z)$ are given by

$$\sinh \pi z = 0$$

$$\frac{e^{\pi z} - e^{-\pi z}}{2} = 0$$

$$e^{2\pi z} = e^{i(2n\pi)} \Rightarrow 2\pi z = 2in\pi$$

$$z = ni \qquad n = 0, \pm 1, \pm 2$$

so $f(z)$ has poles at $z = 0, \pm i, \pm 2i \ldots$

$z = 0, i$ lies on the rectangular contour and are indented to avoid these

Since the integrand is analytic on and within the indented contour, therefore by cauchy fundamental theorem.

$$\int_C f(z)\,dz = 0$$

$$\int_{-R}^{-r_1} f(u)\,du + \int_{C_{r_1}} f(z)\,dz + \int_{r_1}^{R} f(u)\,du + \int_{AB} f(z)\,dz$$

$$+ \int_{R}^{r_2} f(z)\,dz + \int_{C_{r_2}} f(z)\,dz + \int_{-r_2}^{-R} f(z)\,dz$$

$$+ \int_{CD} f(z)\,dz = 0 \longrightarrow (1)$$

Integral Along AB & CD

$$\left|\int_{AB} f(z)\,dz\right| \leq \int_0^1 \left|\frac{e^{a(R+iy)}}{\sinh \pi(R+iy)}\right| |i\,dy|$$

$$\leq \int_0^1 \frac{e^{aR}}{|\sinh \pi(R+iy)|}\,dy$$

$$\leq \int_0^1 \frac{2e^{aR}}{|e^{i\pi(R+iy)} - e^{-i\pi(R+iy)}|}\,dy$$

$$\leq \int_0^1 \frac{2e^{aR}}{|e^{i\pi(R+iy)}| - |e^{-i\pi(R+iy)}|}\,dy$$

$$\leq \frac{2e^{aR}}{e^{\pi R} - e^{-\pi R}} \to 0 \quad \text{as } R \to \infty \text{ fast}$$

Similarly integral along CD is zero

Integral $\int_{C_{r_1}} \frac{e^{az}}{\sinh \pi z}\, dz$

$\lim_{z \to z_0} (z - z_0) f(z)$ = $z(\pi i)$

$\lim_{r \to 0} \int_{C_r} f(z)\, dz$ = $2\pi i$ Res at z_0

$2\pi i$ Res (calculate)

$$\lim_{z \to 0} (z-0) \frac{e^{az}}{\sinh \pi z} \qquad (\tfrac{0}{0})$$

$$= \lim_{z \to 0} \frac{az e^{az} + e^{az}}{\pi \cosh \pi z} = \frac{1}{\pi}$$

$$\lim_{r_1 \to 0} \int_{C_{r_1}} f(z) = -i \cdot \frac{1}{\pi}(\pi) = -i$$

minus sign because direction being clockwise.

Integral Along $\int_{C_{r_2}} \frac{e^{az}}{\sinh \pi z}\, dz$

$$\lim_{z \to i} (z-i) \frac{e^{dz}}{\sinh \pi z} \qquad (\tfrac{0}{0})$$

$$= \lim_{z \to i} \frac{e^{az} + (z-i) a e^{az}}{\pi \cosh \pi z} = \frac{e^{ia}}{\pi \cosh \pi i}$$

$$= \frac{e^{ia}}{\pi \cos \pi} = -\frac{e^{ia}}{\pi} \qquad \therefore \lim_{r_2 \to 0} \int_{C_{r_2}} \frac{e^{az}}{\sinh \pi z} = -i\pi \left(-\frac{e^{ia}}{\pi}\right) = i e^{ia}$$

using these & taking limit when $R \to \infty$ & $r_1, r_2 \to 0$

$$\int_{-\infty}^{0} \frac{e^{ax}}{\sinh \pi x} dx - i + \int_{0}^{\infty} \frac{e^{ax}}{\sinh \pi x} dx + 0 + \int_{\infty}^{0} \frac{e^{a(x+i)}}{\sinh \pi (x+i)} dx$$

$$+ i e^{ia} + \int_{0}^{-\infty} \frac{e^{a(x+i)}}{\sinh \pi (x+i)} dx = 0$$

$$\int_{-\infty}^{0} \frac{e^{ax}}{\sinh \pi x} dx + \int_{0}^{\infty} \frac{e^{ax}}{\sinh \pi x} dx - \int_{0}^{\infty} \frac{e^{ax} \cdot e^{ai}}{-i \sinh i\pi (x+i)} dx$$

$$- \int_{-\infty}^{0} \frac{e^{ax} \cdot e^{ai}}{-i \sin i\pi (x+i)} dx = i - i e^{ia}$$

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{\sinh \pi x}\,dx - \int_{\infty}^{-\infty} \frac{e^{ax}\cdot e^{ia}}{+i \sin i\pi x}\,dx$$

$$-\int_{-\infty}^{0} \frac{e^{ax}\cdot e^{ia}}{+i \sin i\pi x}\,dx = i - ie^{ia}$$

$$\int_{-\infty}^{+\infty} \frac{e^{ax}\,dx}{\sinh \pi x} + \int_{0}^{\infty} \frac{e^{ax}\,e^{ia}\,dx}{\sinh \pi x}$$

$$+ \int_{-\infty}^{0} \frac{e^{ax}\,e^{ia}}{\sinh \pi x}\,dx = i - ie^{ia}$$

$$\int_{-\infty}^{+\infty} \frac{e^{ax}\,dx}{\sinh \pi x} + \int_{-\infty}^{+\infty} \frac{e^{ax}\cdot e^{ia}}{\cosh \pi x} = i - ie^{ia}$$

$$\boxed{(1-e^{ia}) \int_{-\infty}^{+\infty} \frac{e^{ax}}{\sinh \pi x}\,dx = i\left[1 - e^{ia}\right]}$$

$$\int_{-\infty}^{+\infty} \frac{e^{ax}\,dx}{\sin h \pi x} + (\cos a + i \sin a) \int_{-\infty}^{+\infty} \frac{e^{ax}}{\cosh \pi x}\,dx$$

$$= i\left[1 - \cos a - i \sin a\right]$$

$$= i(1 - \cos a) + \sin a$$

Comparing imaginary parts

$$+\sin a \int_{-\infty}^{+\infty} \frac{e^{ax}}{\cosh \pi x}\,dx = 1 - \cos a$$

$$+\int_{-\infty}^{+\infty} \frac{e^{ax}\,dx}{\cosh \pi x} = \frac{1 - \cos a}{\sin a}.$$

$$\int_{-\infty}^{0} \frac{e^{ax}\,dx}{\cosh \pi x} + \int_{0}^{+\infty} \frac{e^{ax}\,dx}{\cosh \pi x} = \frac{2\sin^2(a/2)}{2\sin a/2 \cos a/2}$$

$$+\int_{0}^{\infty} \frac{-e^{-ax}\,dx}{\cosh \pi x} + \int_{0}^{+\infty} \frac{e^{ax}\,dx}{\cosh \pi x} = \tan(a/2)$$

$$\int_{0}^{\infty} \frac{e^{ax} - e^{-ax}}{\cosh \pi x}\,dx = \tan(a/2)$$

$$2\int_{0}^{\infty} \frac{\sinh ax}{\cosh \pi x} = \tan(a/2)$$

$$\int_{0}^{\infty} \frac{\sinh ax}{\cosh \pi x} = \frac{1}{2}\tan(a/2)$$

Example Evaluate $I = \int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1 - e^{x}}\,dx$

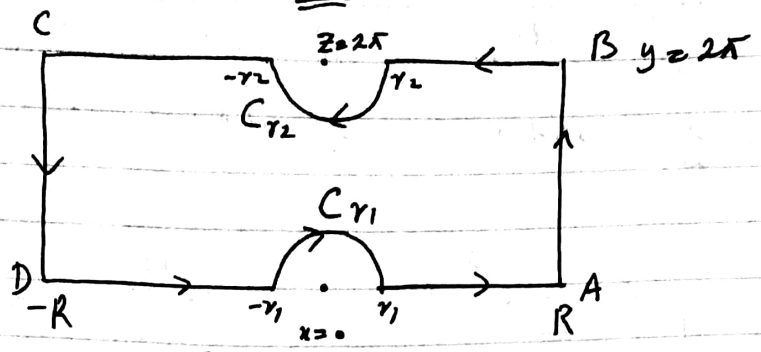
where $0 < p, q < 1$

Sol We observe that this integral is convergent and well defined. We evaluate two separate integrals

$$I_1 = \int_{-\infty}^{+\infty} \frac{e^{px}}{1 - e^{x}}\,dx$$

$$I_2 = \int_{-\infty}^{+\infty} \frac{e^{qx}}{1 - e^{x}}\,dx$$

I_2 $I_1 - I_2$



Consider $J = \oint_C \frac{e^{pz}}{1-e^z}\,dz$

Where C is indented Contour shown above.
Since C encloses no singularities, Therefore

$J = 0$ on top $z = u + 2\pi i$

$$\Rightarrow \left(\int_{-R}^{r_1} + \int_{r_1}^{R}\right)\frac{e^{pu}}{1-e^u}\,du + \left(\int_{AB} + \int_{CD}\right)\frac{e^{pz}}{1-e^z}\,dz$$

$$+\left(\int_{R}^{+r_2} + \int_{-r_2}^{-R}\right)\frac{e^{p(u+2\pi i)}}{1-e^{u+2\pi i}}\,du + \left(\int_{C_{r_1}} + \int_{C_{r_2}}\right)\frac{e^{pz}}{1-e^z}\,dz$$

$$= 0 \longrightarrow (1)$$

Integral $\int_{AB}$ & $\int_{CD}$

$$\left|\int_{AB}\frac{e^{pz}}{1-e^z}\,dz\right| \le \left|\int_0^{2\pi}\frac{e^{p(R+iy)}}{1+e^{R+iy}}\,i\,dy\right|$$

$$\le \frac{e^{pR}}{e^R - 1}\cdot 2\pi \to 0 \quad \text{as } R\to\infty, \; p<1$$

Similarly

$$\int_{CD}\frac{e^{pz}\,dz}{1-e^z} = 0$$

Integral $\int_{C_{r_1}} \frac{e^{pz}}{1-e^z}\,dz$

$$\lim_{z\to 0}(z-0)\cdot\frac{e^{pz}}{1-e^z} \qquad (\tfrac{0}{0})$$

$$= \lim_{z\to 0}\frac{e^{pz}+pze^{pz}}{-e^z} = -1$$

$$\Rightarrow \lim_{r_1\to 0}\int \frac{e^{pz}}{1-e^z}\,dz = -i\pi(-1) = i\pi$$

Integral $\int_{C_{r_2}} \frac{e^{pz}}{1-e^z}\,dz$

$$\text{Res}\left(\frac{e^{pz}}{1-e^z}, 2\pi\right) = \left(\frac{e^{pz}}{-e^z}\right)_{z=2\pi} = -e^{2\pi ip}$$

$$\lim_{r_2\to 0}\int_{C_{r_2}} \frac{e^{pz}\,dz}{1-e^z} = -i\pi\left(-e^{2\pi ip}\right) = i\pi e^{2\pi ip}$$

Hence taking limit when $R\to\infty$, $r_1, r_2 \to 0$

$$\left(\int_{-\infty}^{0} + \int_{0}^{+\infty}\right)\frac{e^{px}}{1-e^x}\,dx + 0 + i\pi + i\pi e^{2\pi ip}$$

$$+\left(\int_{\infty}^{0} + \int_{0}^{-\infty}\right)\frac{e^{2\pi ip}\,e^{px}}{1-e^x}\,dx = 0$$

$$\int_{-\infty}^{+\infty}\frac{e^{px}}{1-e^x}\,dx - \int_{-\infty}^{+\infty}\frac{e^{2\pi ip}\,e^{px}}{1-e^x}\,dx = -i\pi\left(1+e^{2\pi ip}\right)$$

$$\left(1-e^{2\pi ip}\right)\int_{-\infty}^{+\infty}\frac{e^{px}}{1-e^x}\,dx = -i\pi\left(1+e^{2\pi ip}\right)$$

$$\int_{-\infty}^{+\infty}\frac{e^{px}\,dx}{1-e^x} = -\frac{i\pi\left(1+e^{2\pi ip}\right)}{1-e^{2\pi ip}}$$

$I_1 = \pi \cot \pi p$

Similarly

$I_2 = \pi \cot \pi q$

$I_2 \quad I_1 - I_2 = \pi(\cot \pi p - \cot \pi q)$

Example # Using the rectangular contour, prove that

$$\int_0^\infty \frac{t^{\alpha-1}}{1+t}\,dt = \pi \operatorname{cosec} \alpha\pi \qquad 0 < \alpha < 1$$

(for $t = e^x$)

Sol $$I = \int_0^\infty \frac{t^{\alpha-1}}{1+t}\,dt$$

putting $t = e^x$ $\qquad dt = e^x dx$

$\ln t = x$ $\qquad = t\,dx$

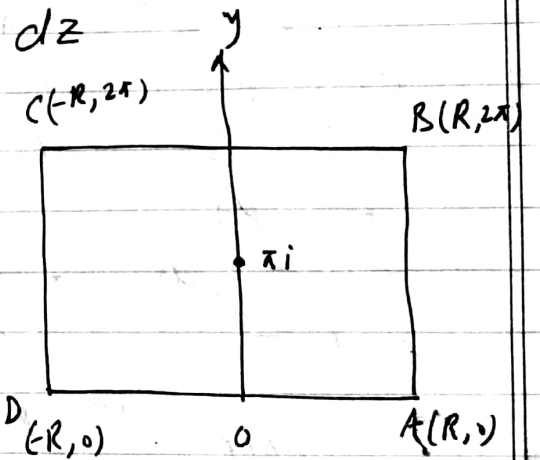
When $t = 0$ $\qquad x = -\infty$

$t = \infty$ $\qquad x = \infty$

$$I = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{e^x + 1}\,dx$$

Consider

$$\int_C f(z)\,dz = \int_C \frac{e^{\alpha z}}{e^z + 1}\,dz$$



Where C is the rectangle ABCD with vertices R, $R + 2\pi i$, $-R + 2\pi i$, $-R$

R being +ve

The poles of $f(z)$ are roots of

$$e^z = -1 = e^{i(\pi + 2n\pi)}$$

$$z = (2n+1)\pi i \qquad \text{where } n = 0, \pm 1, \pm 2$$

only the simple pole $z = \pi i$ (Corresponding $n = 0$) lies inside C

$$\text{Res}(f, \pi i) = \left(\frac{e^{\alpha z}}{\frac{d}{dz}(1+e^z)} \right)_{z=\pi i} = \frac{e^{\alpha \pi i}}{e^{\pi i}} \quad \text{[131]}$$

$$z - e^{\alpha \pi i}$$

$$e^{\pi i} = \cos\pi + i\sin\pi = -1$$

By Cauchy's Residue theorem we have

$$\int_{-R}^{R} f(x)\,dx + \int_{0}^{2\pi} f(R+iy)\,i\,dy + \int_{R}^{-R} f(x+2\pi i)\,dx + \int_{2\pi}^{0} f(-R+iy)\,i\,dy$$

$$= 2\pi i \times \text{sum of residues} = -2\pi i\, e^{\alpha \pi i} \longrightarrow (1)$$

But $$\left| \int_0^{2\pi} f(R+iy)\,i\,dy \right| \le \int_0^{2\pi} |f(R+iy)|\,|i\,dy|$$

$$\le \int_0^{2\pi} \left| \frac{e^{\alpha i(R+iy)}}{1+e^{R+iy}} \right| |dy| \qquad |i| = 1$$

$$\le \int_0^{2\pi} \left| \frac{e^{\alpha R}}{e^R - 1} \right| \le \frac{e^{\alpha R}}{e^R - 1} \cdot 2\pi$$

$$\le \frac{2\pi}{e^{(1-\alpha)R} - e^{-\alpha R}} \to 0 \quad \text{for } 0 < \alpha < 1$$

$$\therefore \int_0^{2\pi} f(R+iy)\,i\,dy \to 0 \quad \text{when } R \to \infty$$

Similarly $\int_0^{2\pi} f(-R+iy)\,i\,dy \to 0$ when $R \to \infty$

provided $0 < \alpha < 1$

Hence when $R \to \infty$

$$\int_{-\infty}^{+\infty} f(x)\,dx + \int_{+\infty}^{-\infty} f(x+2\pi i)\,dx = -2\pi i e^{\alpha \pi i}$$

$$\int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{e^x + 1}\,dx - \int_{-\infty}^{+\infty} \frac{e^{\alpha(x+2\pi i)}}{e^{x+2\pi i} + 1}\,dx = -2\pi i e^{\alpha \pi i}$$

$$\int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{e^x+1}dx - \int_{-\infty}^{+\infty} \frac{e^{\alpha x}\cdot e^{2\alpha\pi i}}{e^x+1}dx = -2\pi i\, e^{\alpha\pi i} \quad \text{[1st]}$$

$$\int_{-\infty}^{+\infty} \frac{e^{\alpha x}\left(1-e^{2\alpha\pi i}\right)}{e^x+1}dx = -2\pi i\, e^{\alpha\pi i}$$

$$\int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{e^x+1} = \frac{-2\pi i\, e^{\alpha\pi i}}{1-e^{2\alpha\pi i}}$$

$$\int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{e^x+1}dx = \pi\cdot\frac{2i}{e^{\alpha\pi i}-e^{-\alpha\pi i}}$$

$$= \frac{\pi}{\left(\frac{e^{\alpha\pi i}-e^{-\alpha\pi i}}{2i}\right)} = \frac{\pi}{\sin\alpha\pi}$$

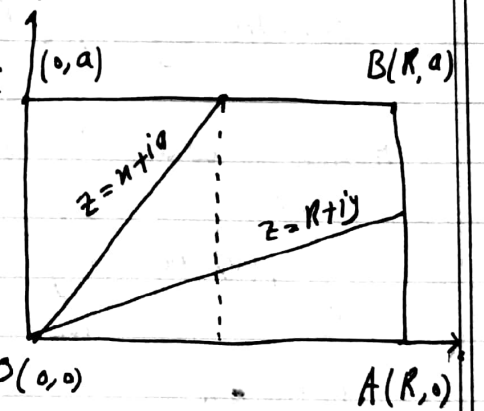
OR

$$\int_0^{\infty} \frac{t^{\alpha-1}}{t+1}dt = \pi\,\mathrm{Cosec}\,\alpha\pi$$

Example# Prove that $\int_0^{\infty} e^{-x^2}\cos 2ax = \frac{1}{2}\sqrt{\pi}\, e^{-a^2}$

by integrating e^{-z^2} round the rectangle whose vertices are $(\pm R, 0)$ $(\pm R\, a)$, $(0, a)$

Sol# Let A be $(R, 0)$ and C be $(0, a)$ in the Argand diagram



on OA $\quad z = x \quad dz = dx$

on AB $\quad z = R+iy \quad dz = i\,dy$

on BC $\quad z = x+ia \quad dz = dx$

on OC $\quad z = iy \quad dz = i\,dy$

Now e^{z^2} has no poles within or on the contour C (consisting of rectangle)

By Cauchy's Theorem

$$\int_C e^{-z^2} dz = 0$$

splitting up the Contour:

$$\int_0^R e^{-x^2} dx + \int_0^a e^{-(R+iy)^2} i\,dy + \int_R^0 e^{-(x+ia)^2} dx + \int_a^0 e^{-(iy)^2} i\,dy = 0 \longrightarrow (1)$$

Now

$$\left| \int_0^a e^{-(R+iy)^2} i\,dy \right| \le \int_0^a \left| e^{-(R^2+2iRy-y^2)} \right| |i|\, dy$$

$$\le \int_0^a e^{-R^2} \cdot e^{y^2} \left| e^{-2iRy} \right| dy$$

$$\because |e^{-2iRy}| = 1$$

$$\le \int_0^a e^{-R^2} e^{y^2} dy \to 0 \text{ as } R \to \infty$$

Hence when $R \to \infty$ (1) reduces to

$$\int_0^\infty e^{-x^2} dx - e^{a^2} \int_0^\infty e^{-(x+ia)^2} dx - i \int_0^a e^{y^2} dy = 0$$

$$\int_0^\infty e^{-x^2} dx - e^{a^2} \int_0^\infty e^{-x^2} [\cos 2ax - i \sin 2ax]\, dx - i \int_0^a e^{y^2} dy = 0$$

Equating real parts

$$e^{a^2} \int_0^\infty e^{-x^2} \cos 2ax\, dx = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty e^{-x^2} \cos 2ax\, dx = \frac{\sqrt{\pi}}{2} e^{-a^2} \quad \text{proved}$$

Proof of $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Let $I = \int_0^\infty e^{-x^2} dx$

$$I^2 = \left(\int_0^\infty e^{-x^2} dx \right)^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$$

Converting to polar form

$x = r\cos\theta \qquad y = r\sin\theta \qquad \tan\theta = y/x$

$dx\,dy = r\,dr\,d\theta$

$$I^2 = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\int_{r=0}^{\infty} e^{-r^2} r\,dr\right] d\theta$$

$$= \int_0^{\pi/2} \cdot -\frac{1}{2}\left[e^{-r^2}\right]_0^{\infty} d\theta = \int_0^{\pi/2} 1/2\, d\theta$$

$$= 1/2\,[\theta]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

$$I = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Example # By integrating the function $\frac{z}{a - e^{-iz}}$ round the rectangular contour with vertices $\pm\pi, \pm\pi + iR$ Show that

$$\int_0^{\pi} \frac{x\sin x\,dx}{a^2 - 2a\cos x + 1} = \begin{cases} \frac{\pi}{a}\log(1+a) & \text{if } 0 < a < 1 \\ \frac{\pi}{a}\log(1+1/a) & \text{if } a > 1 \end{cases}$$

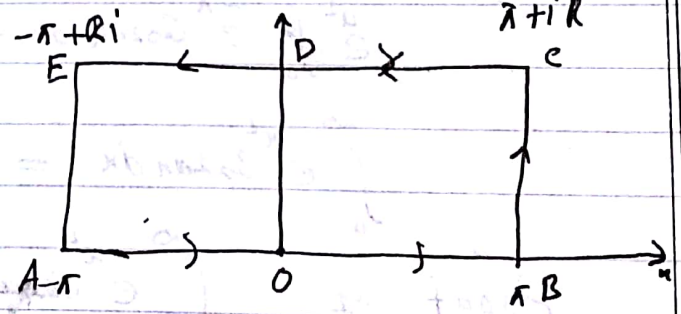
Sol The poles of $\frac{z}{a - e^{-iz}}$ are given by roots of

$$a - e^{-iz} = 0$$

$$a = e^{-iz}$$

$$-iz = \log a$$

$$z = -\frac{1}{i}\log a = i\log a$$



$$R(f, i\log a) = \left[\frac{z\,(z - i\log a)}{a - e^{iz}}\right]_{z = i\log a}$$

$$= \left(\frac{z \cdot 1 + z - i\log a}{i e^{iz}}\right)_{z = i\log a}$$

$$= \frac{i\log a}{ia} = \frac{\log a}{a}$$

$$\int_{AO} f(z)\,dz = \int_{-\pi}^{0} \frac{x\,dx}{a - e^{ix}} = -\int_{0}^{-\pi} \frac{x\,dx}{a - e^{ix}}$$

$$\int_{OB} f(z)\,dz = \int_{0}^{\pi} \frac{x\,dx}{a - e^{-ix}}$$

$$\int_{CD} f(z)\,dz = \int_{R}^{0} \frac{x + iR}{a - e^{-i(x+iR)}}\,dx$$

$$= -\int_{0}^{\pi} \frac{x + iR}{a - e^{-ix} \cdot e^{R}}\,dx \to 0 \quad \text{as } R \to \infty$$

$$\int_{DE} f(z)\,dz = \int_{0}^{-\pi} \frac{x + iR}{a - e^{-ix} e^{R}}\,dx \to 0 \quad \text{as } R \to \infty$$

Case-I Suppose that $a > 1$ so that pole $i\log a$ lies inside the rectangle we have

$$\int_{AO} f(z)\,dz + \int_{OB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{EA} f(z)\,dz$$

$$= 2\pi i \cdot \frac{\log a}{a}$$

$$-\int_{0}^{-\pi} \frac{x\,dx}{a - e^{-ix}} + \int_{0}^{\pi} \frac{x\,dx}{a - e^{-ix}} + \int_{0}^{R} \frac{(\pi + iy)}{a - e^{-i(\pi+iy)}}\,i\,dy$$

$$- \int_{0}^{R} \frac{-\pi + iy}{a - e^{-i(-\pi+iy)}}\,i\,dy = 2\pi i\,\frac{\log a}{a}$$

$$-\int_0^{-\pi}\frac{x\,dx}{a-e^{-ix}}+\int_0^{\pi}\frac{x\,dx}{a-e^{-ix}}+\int_0^{R}\frac{\pi+iy}{a+e^{y}}i\,dy$$

$$+\int_0^{R}\frac{-\pi+iy}{a+e^{y}}i\,dy=2\pi i\cdot\frac{\log a}{a}$$

$$-\int_0^{-\pi}\frac{x\,dx}{a-e^{-ix}}+\int_0^{\pi}\frac{x\,dx}{a-e^{-ix}}+2\pi i\int_0^{R}\frac{dy}{a+e^{y}}=2\pi i\cdot\frac{\log a}{a}$$

putting $x=-t$, $-\int_0^{-\pi}\frac{x\,dx}{a-e^{ix}}$

$$-\int_0^{\pi}\frac{x\,dx}{a-e^{ix}}+\int_0^{\pi}\frac{x\,dx}{a-e^{-ix}}+2\pi i\int_0^{R}\frac{dy}{a+e^{y}}=2\pi i\cdot\frac{\log a}{a}$$

$$\Rightarrow\int_0^{\pi}x\left[\frac{1}{a-e^{-ix}}-\frac{1}{a-e^{ix}}\right]dx+2\pi i\int_0^{R}\frac{dy}{a+e^{y}}=2\pi i\cdot\frac{\log a}{a}$$

$$\Rightarrow\int_0^{\pi}x\left[\frac{a-e^{ix}-(a-e^{-ix})}{(a-e^{-ix})(a-e^{ix})}\right]dx+2\pi i\int_0^{R}\frac{dy}{a+e^{y}}$$

$$=2\pi i\cdot\frac{\log a}{a}$$

$$\Rightarrow\int_0^{\pi}\frac{-2ix\sin x\,dx}{a^2-a(e^{-ix}+e^{ix})+1}+\int=2\pi i\cdot\frac{\log a}{a}$$

$$\Rightarrow\int_0^{\pi}\frac{-2ix\sin x\,dx}{a^2-2a\cos x+1}+2\pi i\int_0^{R}\frac{dy}{a+e^{y}}=2\pi i\cdot\frac{\log a}{a}$$

Taking limit when $R\to\infty$

$$\int_0^{\pi}\frac{-2ix\sin x\,dx}{a^2-2a\cos x+1}+2\pi i\int_0^{\infty}\frac{dy}{a+e^{y}}=2\pi i\cdot\frac{\log a}{a}$$

$$\int_0^{\pi}\frac{-2ix\sin x\,dx}{a^2-2a\cos x+1}+2\pi i\int_0^{\infty}\frac{dy}{a+e^{y}}=2\pi i\cdot\frac{\log a}{a}$$

$$\int_0^\infty \frac{dy}{a+e^y} = \int_0^\infty \frac{\bar{e}^y}{1+a\bar{e}^y}\,dy$$

$$= -\frac{1}{a}\left[\log\left(1+ae^{-y}\right)\right]_0^\infty$$

$$= \frac{\log(1+a)}{a}$$

$$\int_0^\pi \frac{x \sin x\,dx}{a^2 - 2a\cos x + 1} = \frac{2i\pi}{-2i}\left[\frac{\log a}{a} - \frac{\log(1+a)}{a}\right]$$

$$= \frac{\pi}{a}\log\left(\frac{1+a}{a}\right) = \frac{\pi}{a}\log\left(1+\tfrac{1}{a}\right)$$

Example # By integrating $e^{iz} z^{a-1}$ round a quadrant of a circle of radius R, prove that if $0<a<1$

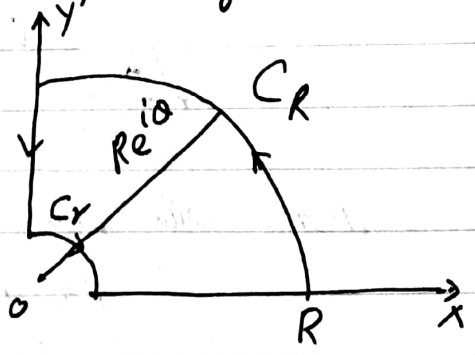
$$\int_0^\infty x^{a-1} \begin{matrix}\cos\\ \sin\end{matrix} x\,dx = \Gamma a \begin{matrix}\cos\\ \sin\end{matrix}\left(\tfrac{\pi}{2}a\right)$$

Sol # Consider $\int_C f(z)dz$, where $f(z) = e^{iz} z^{a-1}$

Choosing the contour C consisting of a quadrant of large circle $|z| = R$ indented at $z=0$. Let r be the radius of identation. By Cauchy's integral, we have

$$\int_C f(z)dz = 0$$

Here $z=\infty$ is not a pole, but $z=0$ is a pole which has been excluded from the contour by identation.



spliting up the above contour we get

$$\int_r^R e^{ix} x^{a-1}\,dx + \int_{C_R} e^{iz} z^{a-1}\,dz + \int_R^r e^{i(iy)} (iy)^{a-1} i\,dy + \int_{C_r} e^{iz} z\,dz = 0$$

$$R(f,0) = \lim_{z\to 0} z \cdot e^{iz} z^{a-1} = 0$$

$$\left| \int_{C_R} e^{iz} z^{a-1} dz \right| = \left| \int_{C_R} e^{iRe^{i\theta}} (Re^{i\theta})^{a-1} iRe^{i\theta} d\theta \right| \qquad z = Re^{i\theta}$$

$$\leq \int_0^{\pi/2} e^{-R\sin\theta} \left| e^{iR\cos\theta} \right| |i| |R^a| |e^{ia\theta}| d\theta$$

$$\leq \int_0^{\pi/2} e^{-R\sin\theta} R^a d\theta$$

$$\leq \int_0^{\pi/2} e^{-2R\theta/\pi} \cdot R^a d\theta \qquad \text{as } \frac{\sin\theta}{\theta} > \frac{2}{\pi}$$

(Jordan's inequality)

$$\leq \frac{R^a}{2} \cdot \frac{\pi}{R} \left[1 - e^{-R}\right] \to 0 \quad \text{as } R \to \infty$$

$$\left| \int_{C_r} e^{iz} z^{a-1} dz \right| \leq \frac{\pi}{2} \cdot \frac{r^a}{r} \left[1 - e^{-r}\right] \qquad \because z = re^{i\theta}$$

$$\leq \frac{\pi}{2} \cdot \frac{r^a}{r} \left(r - \frac{r^2}{2!} + \cdots\right)$$

$$\to 0 \quad \text{as } r \to 0$$

OR

$$\lim_{r\to 0} \int_{C_r} e^{iz} z^{a-1} dz = i \cdot \frac{\pi}{2} \text{ (Residue at } z=0)$$

$$= i \cdot \pi/2 \cdot 0 = 0$$

Hence with these results as $R \to \infty$, $r \to 0$

$$\int_0^\infty e^{ix} x^{a-1} dx - \int_0^\infty e^{-y} \cdot i^a y^{a-1} dy = 0$$

$$\int_0^\infty (\cos x + i \sin x) x^{a-1} dx = i^a \int_0^\infty e^{-y} y^{a-1} dy$$

$$= \left(e^{i\pi/2}\right)^a \lceil a$$

$$\because \int_0^\infty e^{-y} y^{a-1} dy = \lceil a \qquad i = e^{i\pi/2}$$

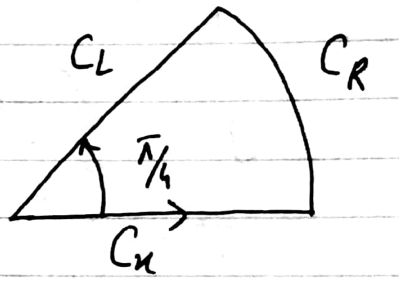
Equating real and imaginary parts

$$\int_0^\infty \cos x \cdot x^{a-1}\,dx = \Gamma a \cos \pi/2\, a$$

$$\& \int_0^\infty \sin x \cdot x^{a-1}\,dx = \Gamma a \sin \pi/2\, a. \quad \text{(proved.)}$$

Example # Evaluate $I = \int_0^\infty e^{it x^2}\,dx$

Sol # For convenience we take $t \to 0$. Consider the Contour depicted in fig where the contour C_R is the sector $Re^{i\theta} : 0 \le \theta \le \pi/4$



Because $e^{it x^2}$ is analytic inside $C = C_x + C_R + C_L$, we have

$$\oint_C e^{it z^2}\,dz = \left(\int_{C_L} + \int_{C_x} + \int_{C_R}\right) e^{it z^2}\,dz = 0$$

$$\left|\int_{C_R} e^{it z^2}\,dz\right| = \left|\int_0^{\pi/4} e^{itR^2(\cos 2\theta + i \sin 2\theta)}\, Re^{i\theta}\, i\,d\theta\right|$$

$$\le \int_0^{\pi/4} R e^{-tR^2 \sin 2\theta}\,d\theta$$

$$\le \int_0^{\pi/4} R e^{-tR^2 \frac{4\theta}{\pi}}\,d\theta = \frac{\pi}{4tR}\left(1 - e^{-tR^2}\right) \to 0 \text{ as } R \to \infty$$

$\because \sin x \ge \frac{2x}{\pi}$ for $0 < x < \pi/2$

$\therefore \sin 2\theta \ge \frac{2(2\theta)}{\pi}$ for $0 < \theta < \pi/4$

Thus $\int_{C_R} e^{it z^2}\,dz \to 0 \quad R \to \infty$

Hence C_x, $z = x$ and on C_L $z = re^{i\pi/4}$

$$\int_{C_R} e^{it z^2} dz + \int_{R}^{0} e^{it(iy)^2} dy = 0$$

$$\int_0^R e^{itx^2} dx + \int_R^0 e^{-ty^2} dy\, e^{i\pi/4}$$

$$\int_0^R e^{itx^2} dx = e^{i\pi/4} \int_0^R e^{-ty^2} dy$$

Taking $R \to \infty$

$$\int_0^\infty e^{itx^2} dx = e^{i\pi/4} \int_0^\infty e^{-ty^2} dy$$

$$J = \int_0^\infty e^{-ty^2} dy$$

$$J^2 = \left(\int_0^\infty e^{-ty^2} dy\right)^2 = \int_0^\infty \int_0^\infty e^{-t(x^2+y^2)} dx\, dy$$

$$= \int_0^{\pi/2} \int_0^\infty e^{-t\rho^2} \rho\, d\rho\, d\theta = \frac{\pi}{4t} \Rightarrow J = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

$$I = \int_0^\infty e^{itx^2} dx = e^{i\pi/4} \cdot \frac{1}{2}\sqrt{\frac{\pi}{t}} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

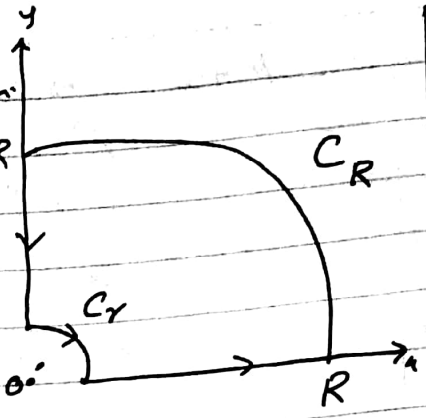
Note Incidently, it should be noted that we can not evaluate the integral I in the same way (via polar co-ordinates) we do on J because I is not absolutely convergent.

Example # Evaluate by contour integration

$$\int_0^\infty \frac{\sin x}{x} dx$$

Sol Choose the contour C as consisting of a the quadrant of a large circle $|z| = R$ with its bounding radii as two axes.

Indented at $z=0$ as shown. Let r be the radius of indentation at $z=0$.



$f(z)$ has the only pole $z=0$, of order one and it does not lie inside the contour C indented at $z=0$.

Cauchy's theorem gives

$$\int_r^R f(x)\,dx + \int_{C_R} f(z)\,dz + \int_R^r f(iy)(i\,dy) + \int_{C_r} f(z)\,dz = 0 \longrightarrow (1)$$

Here

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f(z)||dz| = \int_{C_R} \frac{|e^{iz}||dz|}{|z|}$$

$$\le \int_0^{\pi/2} \frac{e^{-R\sin\theta}}{R} R\,d\theta \qquad z = Re^{i\theta},\ |dz| = R\,d\theta$$

$$\le \frac{1}{R}\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta \qquad \text{if } 0 \le \theta \le \pi/2,\ \sin\theta \ge \frac{2\theta}{\pi}$$

$$\le \frac{1}{R}\left[\frac{e^{-2R\theta/\pi}}{-\frac{2R}{\pi}}\right]_0^{\pi/2}$$

$$\le \frac{-\pi}{2R}\left[e^{-R} - 1\right]$$

$$\le \frac{\pi}{2R}\left[1 - e^{-R}\right] \to 0 \text{ as } R \to \infty$$

$$\left(R(f, 0) = \lim_{z\to 0} \frac{z e^{iz}}{z} = \lim_{z\to 0} e^{iz} = 0\right.$$

$$\therefore \lim_{r\to 0} \int_{C_r} f(z)\,dz = -i\cdot\frac{\pi}{2}\,\text{Res}(f, 0) = -i\cdot\frac{\pi}{2}\cdot 0 = 0$$

-ve sign being taken as C_r is traversed in clockwise sense.

Hence as $R \to \infty$, $r \to 0$ ① reduces to

$$\int_0^\infty f(x)\,dx + \int_\infty^0 f(iy)\,i\,dy = 0$$

$$\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\,dx + \int_0^\infty \frac{e^{i(iy)}}{\sqrt{iy}}\,i\,dy = 0$$

$$\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\,dx = \sqrt{i}\int_0^\infty \frac{e^{-y}}{\sqrt{y}}\,dy$$

Let $\sqrt{y} = t$

$\frac{1}{2} y^{-1/2} dy = dt$

$\frac{dy}{\sqrt{y}} = 2dt$

$$\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\,dx = 2\sqrt{i}\int_0^\infty e^{-t^2}\,dt$$

$$= 2\sqrt{i}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi}\sqrt{i} = \sqrt{\pi}\left[e^{i\pi/2}\right]^{1/2}$$

$$= \sqrt{\pi}\left(\cos\pi/4 + i\sin\pi/4\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}}(1+i)$$

Equating real & imaginary parts

$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \frac{\sqrt{\pi}}{2}$ & $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \frac{\sqrt{\pi}}{2}$

Note Here $I = \int_0^\infty e^{-t^2}\,dt = \int_0^\infty e^{-x^2}\,dx$

$$I^2 = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-x^2}dx\right)$$

$$= \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)$$

$$= \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

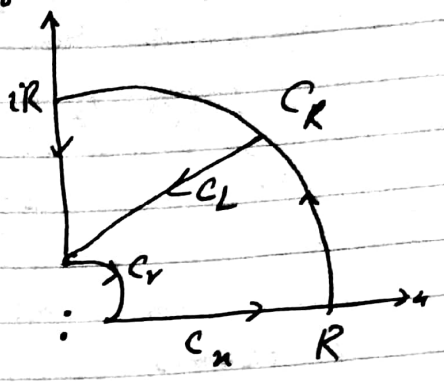
$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\frac{e^{-r^2}}{2}\right]_0^\infty d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}\left[\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\pi/2 - 0\right] = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2}$$

Example # prove that $\int_0^\infty \frac{\sin x^2}{x} = \frac{\pi}{4}$

Sol # Consider $\int_C \frac{e^{iz^2}}{z} dz$



where C is a contour consisting of one quadrant of a large circle $|z| = R$ indented at $z=0$. Let r be the radius of indentation.

By Cauchy's Theorem

$$\int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_R^r f(re^{i\pi/4})(e^{i\pi/4}dy) + \int_{C_r} f(z)dz = 0 \longrightarrow (1)$$

Here $\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} |f(z)||dz| = \int_{C_R} \frac{|e^{iz^2}|}{|z|}|dz|$

$$\le \int_0^{\pi/4} \frac{|e^{iR^2e^{2i\theta}}|}{R} R\,d\theta$$

$$\le \int_0^{\pi/4} \frac{|e^{iR^2\cos 2\theta}||e^{-R^2\sin 2\theta}|}{R} R\,d\theta$$

$$\le \int_0^{\pi/4} \frac{e^{-R^2\sin 2\theta}}{R} \cdot R\,d\theta$$

$$\le \int_0^{\pi/4} e^{-R^2\cdot\frac{4\theta}{\pi}} d\theta \qquad 0 \le \theta \le \pi/4$$

$$\le -\frac{\pi}{4R^2}\left[e^{-\frac{R^2 4\theta}{\pi}}\right]_0^{\pi/4} \qquad \sin\theta \ge \frac{2\theta}{\pi},\ \sin 2\theta \ge \frac{2(2\theta)}{\pi}$$

$$\le -\frac{\pi}{4R^2}\left[\frac{1}{e^{R^2}} - 1\right] \to 0 \quad \text{as } R \to \infty$$

Similarly $\lim \int_{C_2} \frac{e^{iz^2}}{z} dz \to 0$ (142)

$$\int_{C_r} f(z)\,dz = -i \cdot \pi/4 \; Re(f, 0)$$

$$= -i \cdot \pi/4 \left[\frac{e^{iz^2}}{\frac{dz}{dz}}\right]_{z=0}$$

$$= -i\,\pi/4$$

① becomes

$$\int_{\gamma}^{R} \frac{e^{ix^2}}{x} dx - i\,\pi/4 = 0$$

$$\int_{\gamma}^{R} \frac{e^{ix^2}}{x} dx = i \cdot \pi/4$$

Taking limit $\gamma \to 0 \quad R \to \infty$

$$\int_0^{\infty} \frac{e^{ix^2}}{x} dx = i \cdot \pi/4$$

$$\int_0^{\infty} \frac{\cos x^2 + i \sin x^2}{x} dx = i \cdot \pi/4$$

Comparing imaginary parts

$$\int_0^{\infty} \frac{\sin x^2}{x} dx = \pi/4$$

Preparation To Prove The Argument Principle

Before we prove the Argument Principle, it seems suitable to discuss some concepts and details to be used.

Logrithmic Derivative

Let $f(z)$ be analytic in a domain D. Then

$$\frac{f'(z)}{f(z)} \longrightarrow (1)$$

is analytic in D except at the zeros of $f(z)$. If an analytic branch of $\log f(z)$ is chosen in part of D [necessarily excluding the zeros of $f(z)$], then

$$\frac{d}{dz}[\log f(z)] = \frac{f'(z)}{f(z)}$$

For this reason the expression in (1) is called logrithmic derivative. Its value will be demonstrated by Argument principle.

Zeros of Holomorphic Function & Logrithic Derivatives

We return to the study of zeros of a holomorphic function. To take the simplest case 1st. Let f be holomorphic on a region R and suppose f has only one zero at $z=a$ in R of multiplicity m. Then there is a holomorphic function g on R without any zeros such that

$$f(z) = (z-a)^m g(z) \qquad \forall z \in R$$

$$f'(z) = m(z-a)^{m-1} g(z) + (z-a)^m g'(z)$$

$$\frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1} g(z) + (z-a)^m g'(z)}{(z-a)^m g(z)}$$

$$= \frac{m}{z-a} + \frac{g'(z)}{g(z)}$$

Let C be any closed, piecewise smooth path in R not passing through a.

Since $\frac{g'}{g}$ is holomorphic on R,

$$\int_C \frac{g'}{g}\, dz = 0$$

and $$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz = \frac{m}{2\pi i}\int_C \frac{dz}{z-a}$$

$$= m \cdot \frac{1}{2\pi i}\int_C \frac{dz}{z-a}$$

$$= m \, \mathrm{Ind}_C(a)$$

In particular if C is a circle in R and a, is any point inside C, $\mathrm{Ind}_C(a) = 1$ and

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz = m, \text{ multiplicity of zero at } a$$

if a is outside the circle C, then $\mathrm{Ind}_C(a) = 0$ &

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz = 0$$

Thus we see that $\frac{1}{2\pi i}\int_C \frac{f'}{f}\, dz$ = number of zeros of f falling inside the circle, Counting the zero(a) as often as its multiplicity

Suppose f is holomorphic on a region R and f has finitely many zeros say $z_1, z_2 \dots z_k$ of multiplicity $m_1, m_2 \dots m_k$ respectively. Then Using Taylor's expansion of f at z_1, we see that there exists a holomorphic map g_1 such that

$$g_1(z) \neq 0 \text{ and}$$

$$f(z) = (z-z_1)^{m_1} g_1(z) \quad \text{for } z \in R.$$

Factorising $g_1(z)$ Similarly, we get a holomorphic map g_2 such that $g_2(z_2) \neq 0$ and

$$g_1(z) = (z-z_2)^{m_2} g_2(z)$$

So that

$$f(z) = (z-z_1)^{m_1} (z-z_2)^{m_2} g_2(z)$$

Considering the finitely many zeros $z_1, z_2 \dots z_k$ of f in succession thus, we see there exists a holomorphic map g on R which has no zeros in R and

$$f(z) = (z-z_1)^{m_1} (z-z_2)^{m_2} \dots (z-z_k)^{m_k} g(z) \quad \text{for } z \in R$$

Note # An arbitrary holomorphic map need not have only finitely many zeros (e.g $\sin z$). However, note that zeros of a holomorphic map f are isolated ie each zero of f has a neighbourhood which contains no other zero of f. Thus any bounded set in $\mathbb{C}$ contains at most finitely many zeros of f. We use this fact below.

Theorem # Let f be holomorphic on a region D with $z_1, z_2 \dots z_k \dots$ as its zeros with multiplicities $m_1, m_2 \dots m_k \dots$ respectively. Then for any closed path C in D not passing Through any z_j, we have,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{\infty} m_j \cdot \text{Ind}_C(z_j)$$

Proof Let $R>0$ be such that $B(0,R)$ contains portion C^* of C i.e. $C^* \subset B(0,R)$. As noted above, there exist only finitely many zeros in $B(0,R)$ say $z_1, z_2, \ldots, z_k$. obviously

$$Ind_C(z_i) = 0 \quad \text{for } j \geq k+1$$

and

$$\sum_{j=1}^{\infty} m_j \, Ind_C(z_j) = \sum_{j=1}^{k} m_j \, Ind_C(z_j).$$

We may write

$$f(z) = (z-z_1)^{m_1}(z-z_2)^{m_2} \cdots (z-z_k)^{m_k} g(z)$$

where g is holomorphic on D and has no zero in $B(0,R)$. For any $z \in D \cap B(0,R)$ and $z \neq z_j$ $(j = 1, \ldots, k)$ where

$$\frac{f'(z)}{f(z)} = \frac{m_1}{z-z_1} + \frac{m_2}{z-z_2} + \cdots + \frac{m_k}{z-z_k} + \frac{g'(z)}{g(z)}$$

Hence

$$\frac{1}{2\pi i}\int \frac{f'(t)\,dt}{f(t)} = \sum_{j=1}^{k} m_j \left(\frac{1}{2\pi i}\int \frac{dt}{t-z_j}\right) + \frac{1}{2\pi i}\int \frac{g'(t)}{g(t)}\,dt$$

$$= \sum_{j=1}^{k} m_j \, Ind_C(z_j)$$

$$= \sum_{j=1}^{\infty} m_j \, Ind_C(z_j)$$

Corollary 1# For an entire function f and for any circle C at the origin not passing any zero of f, $\frac{1}{2\pi i}\int_C \frac{f'}{f}$ is an integer and equals the no of zeros of f which fall inside C, each zero being counted as often as its multiplicity.

Proof# In this case, $Ind_C(z_j) = 1$ for every z_j in C

$$\frac{1}{2\pi i}\int_C \frac{f'(t)\,dt}{f(t)} \overset{147}{=} k \text{ or } \sum_{j=1}^{k} m_j(1)$$ when $z_j, j=1,\dots k$ lies inside C

$=$ no of zeros inside C, each zero being counted as often as its multiplicity.

Corollary:-2 Let f be entire, let $\omega \in \mathbb{C}$ be given and C' be a circle not passing through any z such that $f(z)=\omega$. Then

$$\frac{1}{2\pi i}\int_{C'} \frac{f'(t)\,dt}{f(t)-\omega}$$ is an integer

and equals the number of times f assumes the value ω inside C'

Proof # let $g(z) = f(z) - \omega$

and let $z_1, z_2 \dots z_k$ be such that

$f(z_j) = \omega \quad j = 1, 2 \dots k$ i.e

$z_1, z_2 \dots z_k$ are zeros of $g(z)$ inside C'. Then by above Corollary

$$\frac{1}{2\pi i}\int_{C'} \frac{g'(t)}{g(t)}\,dt = \text{no of zeros of } g \text{ inside } C'$$

$$= k$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C'} \frac{f'(t)\,dt}{f(t)-\omega} = k = \text{no of times } f \text{ assumes value } \omega \text{ inside } C'$$

Theorem # Let f be a meromorphic function on region R. Then prove that a zero of f of order n is a pole of simple order with residue n and a pole of order p is is a simple pole of $\frac{f'(z)}{f(z)}$ with residue $-p$

Proof # Taking $z=a$ as a zero of order n for $f(z)$

, we have

$$f(z) = (z-a)^n \varphi(z)$$

where $\varphi(z)$ is analytic and not zero at $z=a$

$$\text{Thus } \frac{f'(z)}{f(z)} = \frac{n}{z-a} + \frac{\varphi'(z)}{\varphi(z)}$$

Since $\varphi(z)$ is analytic and so is $\varphi'(z)$, therefore $\frac{\varphi'(z)}{\varphi(z)}$ is analytic at $z=a$. It follows that

$\frac{f'(z)}{f(z)}$ has a simple pole at $z=a$ with residue n

Similarly if $z=b$ is a pole of order p, then there is an analytic function $\psi(z)$ such that

$$f(z) = (z-b)^{-p} \psi(z)$$

$$\text{and } \frac{f'(z)}{f(z)} = \frac{-p}{z-b} + \frac{\psi'(z)}{\psi(z)}$$

It follows that $\frac{f'(z)}{f(z)}$ has a pole of order one at $z=b$ with residue $-p$

<u>Remarks</u> Suppose $z=z_1$ is a zero/pole of $f(z)$ of order n. Then

$$\frac{f'(z)}{f(z)} = \frac{\pm n}{z-z_1} + \varphi(z)$$

where $\varphi(z)$ is analytic in the neighborhood z and the plus/minus sign stands for the zero / pole case respectively as we have seen above

Theorem# (Argument Principle)

Let $f(z)$ be analytic in domain D. Let C be a a simple closed contour in D within which $f(z)$ is analytic except for a finite number of poles and has no zeros on C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N_0 - N_p$$

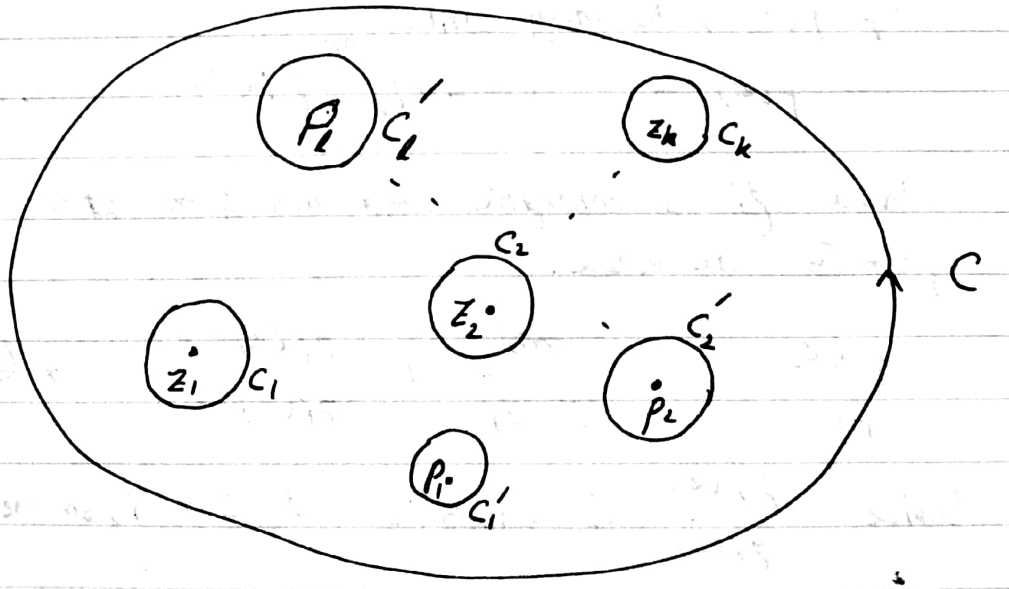
$$= \frac{1}{2\pi}\Delta_C \arg f(z)$$

$$= \frac{1}{2\pi}\left[\arg f(z)\right]_C$$

Δ_C (Delta Sub C)

where ΔC denotes variation of $\arg f(z)$ round the closed contour C, N_0 is total no of Zeros of f inside C and N_p is total number of poles of f inside C, Zeros and poles being counted according to their multiplicities.

Proof# Let $z_1, z_2 ---- z_k$ be zeros of $f(z)$ inside C of multiplicities $m_1, m_2 ---- m_k$ resptly and $p_1, p_2 ---- p_l$ be poles of $f(z)$ inside C of orders $n_1, n_2 ---- n_l$ respectively.



We enclose each zero and pole by non-overlapping circles $C_1, C_2 \ldots C_k$ and $C_1', C_2' \ldots C_l'$ respectively each of radii r.

Since zeros and poles are isolated, it is possible to find such a r.

Now function $\frac{f'(z)}{f(z)}$ has poles $z_1, z_2 \ldots z_k$ and $p_1, p_2 \ldots p_l$ of order one each. So the region above is a multiply-connected region for function $f'(z)/f(z)$ and we have.

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{C_2}\frac{f'(z)}{f(z)}dz + \cdots$$

$$\cdots + \frac{1}{2\pi i}\int_{C_k}\frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{C_1'}\frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{C_2'}\frac{f'(z)}{f(z)}dz + \cdots \frac{1}{2\pi i}\int_{C_l'}\frac{f'(z)}{f(z)}dz$$

$$= \sum_{i=1}^{k}\frac{1}{2\pi i}\int_{C_i}\frac{f'(z)}{f(z)}dz + \sum_{i=1}^{l}\frac{1}{2\pi i}\int_{C_i'}\frac{f'(z)}{f(z)}dz \longrightarrow (1)$$

Since each z_i, is a zero of order m_i $i=1,\ldots k$ of $f(z)$, we may write

$$f(z) = (z - z_i)^{m_i} g_i(z) \longrightarrow (2)$$

where g_i is analytic and non-zero at $z = z_i$

from (2) we have.

$$\frac{f'(z)}{f(z)} = \frac{m_i}{z - z_i} + \frac{g_i'(z)}{g_i(z)} \longrightarrow (3)$$

Since $\frac{g_i'(z)}{g_i(z)}$ is analytic at $z = z_i$, so we have

$$\int_{C_i} \frac{g_i'(z)}{g_i(z)} dz = 0 \qquad \text{(using C.F Theorem)}$$

Also $$\int_{C_i} \frac{m_i\, dz}{z - z_i} = m_i \int_0^{2\pi} \frac{i\,\ell\, e^{i\theta} d\theta}{\ell e^{i\theta}}$$

$$= i m_i \int_0^{2\pi} d\theta = i m_i (2\pi)$$

$$= 2\pi i\, m_i$$

Using relations ① & ③, we get

$$\sum_{i=1}^{k} \frac{1}{2\pi i} \int_{C_i} \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{k} \frac{1}{2\pi i} 2\pi i\, m_i$$

$$= \sum_{i=1}^{k} m_i \longrightarrow ④$$

$= N_0$ = the total no of zeros of f, counted according to multiplicities.

Also since p_i is a pole of order n_i of $f(z)$, we have

$$f(z) = \frac{h(z)}{(z - p_i)^{n_i}}$$

where $h(z)$ is analytic and non-zero at $z = p_i$.

we have

$$\frac{f'(z)}{f(z)} = -\frac{n_i}{z - p_i} + \frac{h'(z)}{h(z)} \longrightarrow ⑤$$

Since $\frac{h'(z)}{h(z)}$ is analytic at $z = p_i$, we have

$$\int_{C_i'} \frac{h'(z)}{h(z)} dz = 0 \qquad \text{(using C.F Theorem)}$$

Again

$$\int_{C_i'} -\frac{n_i}{z - p_i} dz = -n_i \int_0^{2\pi} \frac{i r e^{i\theta} d\theta}{r e^{i\theta}} = -n_i (2\pi i)$$

using relations ① & ⑤

$$\sum_{i=1}^{l} \frac{1}{2\pi i} \int_{C_i'} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \sum_{i=1}^{l} (a - 2\pi i n_i)$$

$$= -\sum_{i=1}^{l} n_i$$

$$= -N_p \quad \rightarrow ⑥$$

where N_p = total no of poles of f counted according to the multiplicities (orders)

using ④ & ⑥ in ①

$$I = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{k} m_i - \sum_{i=1}^{l} n_i = N_0 - N_p \quad \rightarrow ⑦$$

Proof of $I = \frac{1}{2\pi} [\arg f(z)]_C$

In order to show that

$$I = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi} [\arg f(z)]_C$$

we parametrize C as under

let $z = z(t)$ on $a \le t \le b$

where $z(a) = z(b)$ because C is closed

Thus we have for I, the line integral

$$I = \frac{1}{2\pi i} \int_a^b \frac{f'(z(t))}{f(z(t))} z'(t)\, dt = \frac{1}{2\pi i} \left[\log f(z(t)) \right]_{t=a}^{b}$$

$$= \frac{1}{2\pi i} \left[\log |f(z(t))| + i \arg f(z(t)) \right]_{t=a}^{b}$$

$$= \frac{1}{2\pi i} \left[\log |f(z(b))| + i \arg f(z(b)) - \log |f(z(a))| - i \arg f(z(a)) \right]$$

$$= \frac{1}{2\pi i}\left[i\{ \arg f(z(b)) - \arg f(z(a)) \right] \quad \because z(a) = z(b)$$

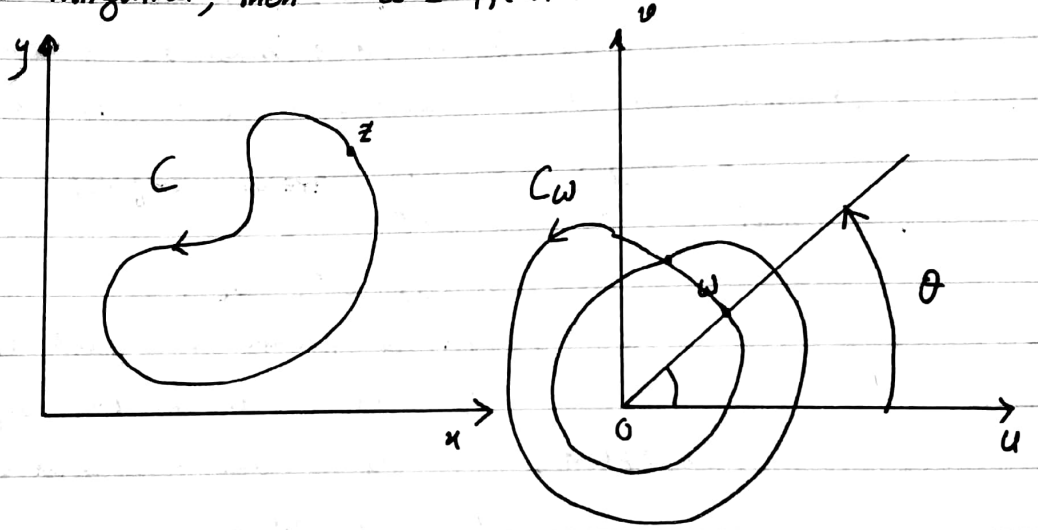
$$= \frac{1}{2\pi}\left[\arg f(z(b)) - \arg f(z(a)) \right]$$

$$= \frac{1}{2\pi}\left[\arg f(z)\right]_C = \frac{1}{2\pi}\,\Delta_C \arg f(z) \longrightarrow (8)$$

Where we have taken principal branch of the logarithm. Also note that $f(z(b)) = f(z(a))$ because $z(a) = z(b)$ but two equal complex numbers may have different argument and hence $[\arg f(z(b)) - \arg f(z(a))]$ is variation of argument $f(z)$ when z varies on contour C

Geometrically Eq (8) corresponds to the following

Let $\omega = f(z)$ be the image of point z under the mapping $\omega = f(z)$ and let $\theta = \arg f(z)$ be the angle that ray from origin to ω makes with respect the horizontal; then $\omega = |f(z)|\, e^{i\theta}$



As z traces the path C, the point $\omega = f(z)$ traces a path C_ω in the ω plane. The path C_ω will be a closed path, but it may cross itself many times (i.e may not be simple). By assumption, $f(z) \neq 0$ on C, so that C_ω does not pass through the origin of the ω-plane

under the transformation $\omega = f(z)$ we find

$$I = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_\omega} \frac{d\omega}{\omega} = \frac{1}{2\pi i}\left[\arg \omega\right]_{C_\omega} \longrightarrow (9)$$

Now the integral

$$\int_{C_w}^{w_2} \frac{dw}{w} = \log w_2 - \log w_1$$

measures the total change in $\log w$ as $\log w$ varies continuously on the path C_w

If $w_1 = w_2$

$$\log w_2 - \log w_1 = \log|w_2| + i \arg w_2 - \log|w_1| - i \arg w_1$$

$$= i(\arg w_2 - \arg w_1) \quad \because w_1 = w_2$$

Hence on the closed path C_w, the integral

$$\int_{C_w} \frac{dw}{w}$$

is pure imaginary and measures i times the total change in $\arg w$ as $\arg w$ varies continuously on the path. This total change in argument must be multiple of 2π, since $w_1 = w_2$ i.e C_w is closed. Hence the integral

$$\frac{1}{2\pi i} \int_{C_w} \frac{dw}{w} = \frac{1}{2\pi}(\arg w_2 - \arg w_1)$$

$$= \frac{1}{2\pi i} (\arg w)_{C_w}$$

= the no of times the point w (or path C_w) winds around the origin in w plane, or image curve C_w when z moves round C

The quantity $\frac{1}{2\pi i}[\arg w]_{C_w}$ is called winding number of C_w about the origin.

Thus

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi} [\text{increase in } \arg f(z) \text{ on path}]$$

$$= N_0 - N_p = \frac{1}{2\pi i} \int_{C_w} \frac{dw}{w}$$

The statement $\frac{1}{2\pi}[\text{increase in } \arg f(z) \text{ on path}]$

$= N_0 - N_p$ is called argument

principle.

Remarks # When $f(z)$ is analytic within C. then $N_p = 0$ i.e no pole in C and we have

$$\frac{1}{2\pi}[\arg f(z)]_C = N_0 - N_p = N_0 - 0 = N_0$$

$$\frac{1}{2\pi}[\text{increase in } \arg f(z) \text{ on the path}] = N_0$$

Some authors call it argument Principle.

This is of great value in finding roots of analytic functions.

General Versions of Argument Principle

Now we state and prove two slightly more general versions of the principle where condition of simple closed path is relaxed.

Theorem # Let f be meromorphic on domain D. Let $z_1, z_2 \ldots\ldots z_k$ be its zeros of multiplicities $m_1, m_2 \ldots\ldots m_k$ respectively and $p_1, p_2 \ldots\ldots p_l$ be its poles of orders $n_1, n_2 \ldots\ldots n_l$ respectively. Let C be any closed path in D not passing through any z_j or p_j. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)\,dz}{f(z)} = \sum_{j=1}^{k} m_j \cdot \text{Ind}_C(z_j) - \sum_{j=1}^{l} n_j \cdot \text{Ind}_C(p_j)$$

Proof # Let g be a holomorphic function on D such that

$$f(z) = \frac{(z-z_1)^{m_1}(z-z_2)^{m_2}\ldots\ldots(z-z_k)^{m_k}}{(z-p_1)^{n_1}(z-p_2)^{n_2}\ldots\ldots(z-p_l)^{n_l}}\, g(z)$$

and $g(z_j) \neq 0$ for every j. Then g is nowhere zero on D

and for $z \neq z_j$

$$\log f(z) = m_1 \log(z-z_1) + m_2 \log(z-z_2) + \cdots + m_k \log(z-z_k) + \log g(z) - n_1 \log(z-p_1) - n_2 \log(z-p_2) - \cdots - n_l \log(z-p_l)$$

Diff.

$$\frac{f'(z)}{f(z)} = \frac{m_1}{z-z_1} + \frac{m_2}{z-z_2} + \cdots + \frac{m_k}{z-z_k} - \frac{n_1}{z-p_1} - \frac{n_2}{z-p_2} - \cdots - \frac{n_l}{z-p_l} + \frac{g'(z)}{g(z)}$$

$$= \sum_{j=1}^{k} \frac{m_j}{z-z_j} - \sum_{j=1}^{l} \frac{n_j}{z-p_j} + \frac{g'(z)}{g(z)}$$

By hypothesis $z_j \notin C$, $p_j \notin C$ for each j, so $\mathrm{Ind}_C(z_j)$ and $\mathrm{Ind}_C(p_j)$ are well defined. Since g'/g is holomorphic on D,

$$\int_C \frac{g'}{g}\, dz = 0 \quad \text{Hence}$$

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \sum_{j=1}^{k} m_j \int_C \frac{dz}{z-z_j} - \frac{1}{2\pi i} \sum_{j=1}^{l} n_j \int_C \frac{dz}{z-p_j} + \frac{1}{2\pi i} \int_C \frac{g'(z)}{g(z)} dz$$

$$= \sum_{j=1}^{k} m_j \, \mathrm{Ind}_C(z_j) - \sum_{j=1}^{l} n_j \, \mathrm{Ind}_C(p_j)$$

If $\gamma : [a\ b] \to \mathbb{C}$ is a path in $\mathbb{C}$, then image $\gamma([a\ b])$ is denoted by γ^*. Since a path is continuous function on closed interval $[a, b]$, γ^* is compact, connected set in $\mathbb{C}$

Remarks # An arbitrary holomorphic map need not have only finitely many zeros (e.g. $\sin z$) and so the above form of Theorem is not more general. We note that zeros of

zeros of f are isolated ie each zero of f has a neighbourhood which contains no other zero. Thus any bounded set in $\mathbb{C}$ contains at most finitely many zeros of. We use this fact to prove the more general form of argument Principle.

Theorem # (Argument Principle in More General Form)

Let f be meromorphic over $\mathbb{C}$ (complex plane) and let $z_1, z_2 \ldots z_k \ldots$ be its zeros of multiplicities $m_1, m_2 \ldots m_k \ldots$ respectively and $p_1, p_2 \ldots p_l \ldots$ be its poles of orders $n_1, n_2 \ldots n_l \ldots$ respectively. For any closed path C not passing through any z_j and p_j

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{j=1}^{\infty} m_j \operatorname{Ind}_C(z_j) - \sum_{j=1}^{\infty} n_j \operatorname{Ind}_C(p_j)$$

Proof # Since C^* (image set of $C:[a,b]\to\mathbb{C}$) is compact, there is $R>0$ such that

$$C^* \subset B(0\ R).$$

For any z_j or p_j such that $|z_j|>R$ or $|p_j|>R$, the zeros or pole belong to unbounded component $\mathbb{C}\setminus C^*$, so Ind_C at that point is zero. Hence both $\sum_{j=1}^{\infty} m_j \operatorname{Ind}_C(z_j)$ and $\sum_{j=1}^{\infty} n_j \operatorname{Ind}_C(p_j)$ involve only finitely many non-zero terms and thus are well defined. Let $z_1, z_2 \ldots z_k$ and $p_1, p_2 \ldots p_l$ be zeros and poles respectively of f lying in $B(0,R)$. So

$$f(z) = \frac{(z-z_1)^{m_1}(z-z_2)^{m_2}\ldots(z-z_k)^{m_k}}{(z-p_1)^{n_1}(z-p_2)^{n_2}\ldots(z-p_l)^{n_l}}\, g(z)$$

where g is holomorphic and non-zero on $B(0,R)$

Proceeding as in previous Theorem

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{k} m_j \operatorname{Ind}_C(z_j) - \sum_{j=1}^{l} n_j \operatorname{Ind}_C(p_j)$$

$$= \sum_{j=1}^{\infty} m_j \operatorname{Ind}_C(z_j) - \sum_{j=1}^{\infty} n_j \operatorname{Ind}_C(p_j)$$

Proved

Corollary # Let f be as above and C be any cicle not passing Through z_j and p_j, Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N_0 - N_p \text{ where}$$

N is the no of zeros of f lying inside C, each zero being Counted as often as its multiplicity and N_p is the number of poles of f lying inside C, each pole being Counted as often as its order

Proof # We have

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{\infty} m_j \operatorname{Ind}_C(z_j) - \sum_{j=1}^{\infty} n_j \operatorname{Ind}_C(p_j)$$

Let $z_1, z_2 \ldots z_k$ be zeros inside C and $p_1, p_2 \ldots p_l$ be poles inside C. Then

$$\operatorname{Ind}_C(z_j) = 1 \quad j = 1, 2 \ldots k$$
$$= 0 \quad j = k+1, k+2 \ldots$$

$$\operatorname{Ind}_C(p_j) = 1 \quad j = 1, 2 \ldots l$$
$$= 0 \quad j = l+1, l+2 \ldots$$

Hence

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{k} m_j - \sum_{j=1}^{l} n_j = N_0 - N_p$$

Extension of Argument Principle

The following extension of argument principle (ef) is useful.

Theorem # Let f be meromorphic on a domain D and let $z_1, z_2 \ldots z_k$ of multiplicities $m_1, m_2 \ldots m_k$ and $p_1, p_2 \ldots p_l$ of orders $n_1, n_2 \ldots n_l$ be its zeros and poles respectively. Let $\varphi(z)$ be any holomorphic function on D. Then for any closed path C not passing through z_j and p_j

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\varphi(z)dz = \sum_{j=1}^{k} m_j \, Ind_C(z_j)\,\varphi(z_j) - \sum_{j=1}^{l} n_j \, Ind_C(p_j)\,\varphi(p_j)$$

Proof # From the basic Equation.

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{k}\frac{m_j}{z-z_j} - \sum_{j=1}^{l}\frac{n_j}{z-p_j} + \frac{g'(z)}{g(z)}$$

we get for any $z \neq z_j$

$$\frac{f'(z)\,\varphi(z)}{f(z)} = \sum_{j=1}^{k}\frac{m_j\,\varphi(z)}{z-z_j} - \sum_{j=1}^{l}\frac{n_j\,\varphi(z)}{z-p_j} + \frac{g'(z)\varphi(z)}{g(z)}$$

where the last term on R.H.S is a holomorphic function on D whose integral over C is zero. We have

$$\int_C \frac{f'(z)\,\varphi(z)}{f'(z)}dz = \sum_{j=1}^{k} m_i \int_C \frac{\varphi(z)}{z-z_j} - \sum_{j=1}^{l} n_j \int_C \frac{\varphi(z)}{z-p_j}$$

By Cauchy's generalized integral formula we have $\int_C \frac{f(z)\,dz}{z-a} = Ind_C(a)\cdot f(a)$ so

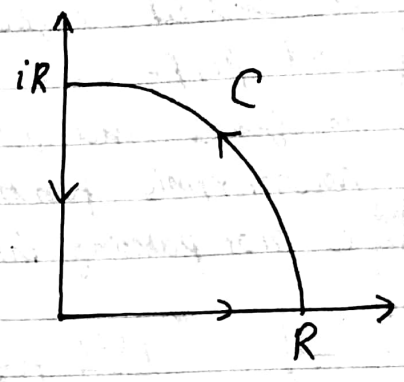
$$\int_C \frac{\varphi(z)f'(z)\,dz}{f(z)} = \sum_{j=1}^{k} m_i \, Ind_C(z_j)\,\varphi(z_j) - \sum_{j=1}^{l} n_j \, Ind_C(p_j)\,\varphi(p_j)$$

proved

Example # Determine the number of zeros located inside (the number) the 1st quadrant of function

$$f(z) = z^3 + 1$$

Sol Consider the Contour in fig



we begin at $z=0$, where.

$f(z) = 1$. We take

$\arg f = \varphi =$

$$\tan\varphi = \frac{Im\, f(x,y)}{Re\, f(x,y)}$$

with the principal brand. $\varphi = 0$

Corresponding to $f(z) = 1$. On the circle $|z| = R$

$$z = Re^{i\theta}$$

$$f(z) = R^3 e^{3i\theta}\left(1 + \frac{1}{R^3 e^{3i\theta}}\right)$$

on the real axis $z = x$, $f(x,y) = x^3 + 1$ is true and continuous. hence at $x = R$ $y = 0$ $\theta(R,0) = 0$

Next we follows the $\arg f(x,y)$ around the circular Contour.

For Large R, $1 + \frac{1}{R^3 e^{3i\theta}} \approx 1$

So

$$f(z) = R^3 e^{3i\theta}$$

hence $\arg f(z) \approx 3\theta$

At $z = iR$, $\theta = \pi/2$; hence $\arg f(z) \approx \frac{3\pi}{2}$

On the imaginary axis $z = iy$

$$f(x,y) = -iy^3 + 1$$

as $y \to$ traverses from $y \to -\infty$ to $y \to 0^+$, $\tan\varphi$ goes from $-\infty$ to 0^- (note $\tan\varphi \approx -y^3$ for $y \to \infty$) i.e φ goes from $3\pi/2$ to 2π. Note also that we are in the